11.2 Events involving '~~Exd~~ Not' and 'OR'

Recall

Theoretical probability

Consider an event E

$$\boxed{P(E) = \frac{n(E)}{n(S)}} \quad \text{— } (*)$$

The formula $(*)$ is valid when all the outcomes in S are equally likely

We know

$$0 \le n(E) \le n(S) \quad \text{— } (1)$$

divide (1) by $n(S)$ (This is a positive integer)

$$\frac{0}{n(S)} \le \frac{n(E)}{n(S)} \le \frac{n(S)}{n(S)}$$

$$0 \le P(E) \le 1$$

Properties of probability

Let E be an event within a sample space S $(E \subseteq S)$ (E is a subset of S)

The following holds

1. $0 \leq P(E) \leq 1$

2. $P(\phi) = 0$

3. $P(S) = 1$

Consider # 29 from Exam #2
(10.5)
#29 If a license plate consists of six digits, how may different licenses could be created having at least one digit repeated

A = at least one digit repeated
A' = no digit repeated

$$\boxed{n(A) = n(\mathcal{U}) - n(A')}$$

$$= 10^6 - \boxed{10} \cdot \boxed{9} \cdot \boxed{8} \cdot \boxed{7} \cdot \boxed{6} \cdot \boxed{5}$$

$$= 10^6 - 151200$$

$$= 848800$$

Exercise

when a Single card is drawn from a 52- card deck, what is the probability it will not be a king

$E =$ not a king

$E' =$ it is a king

$$P(E) = 1 - P(E')$$

$$= 1 - \frac{n(E')}{n(S)} = 1 - \frac{4}{52} = \frac{52-4}{52}$$

$$= \frac{48}{52} = \frac{12}{13}$$

Events involving 'OR'

Recall from 10.5

a counting exercise involving 'or'

$$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$$

when A, B are mutually exclusive

$$n(A \text{ or } B) = n(A) + n(B)$$

Consider Question #27 from Exam #2

(10.5)
If a single card is drawn from a standard 52-card deck, in how many ways could it be an ace or a spade

$$n(\text{Ace or Spade}) = n(\text{Ace}) + n(\text{Spade}) - n(\text{ace and Spade})$$

$$= 4 + 13 - 1$$

$$= 16$$

Example 2

If 4 fair coins are tossed, find the probability of getting at least 2 heads

$n(S) = 2^4 = 16$

E = at least two heads

$P(E) = 1 - P(E')$

E' = not at least two heads
(no heads or 1)

tttt, ttth, ttht, thtt, httt

$P(E) = 1 - \frac{5}{16} = \frac{11}{16}$

11.3 Conditional probability and Events involving 'AND'

Example

(a) Suppose there are 11 balls, find prob she gets a red, yellow & blue ball

5 yellow balls
2 blue balls
4 Red balls

$$P(R_1 \cap Y_2 \cap B_3) = P(R_1) \cdot P(Y_2 | R_1) \cdot P(B_3 | R_1 \cap Y_2)$$

$$= \frac{4}{11} \cdot \frac{5}{10} \cdot \frac{2}{9}$$

(b) Suppose five cards are drawn without replacement from a standard 52-card deck find prob they are all hearts

approach 1

$P(\text{all hearts}) =$

$$P(H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5) = P(H_1) \cdot P(H_2 | H_1) \cdot P(H_3 | (H_1 \cap H_2)) \cdot P(H_4 | (H_1 \cap H_2 \cap H_3)) \cdot P(H_5 | (H_1 \cap H_2 \cap H_3 \cap H_4))$$

$$= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$$

approach 2

$$\frac{13C5}{52C5}$$

Conditional probability

Probability of event B given that A has happened

conditional probability of B given A

$P(B|A)$

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Independent Events

Two events are independent if the knowledge about the occurrence of one has no effect on the other one occurring

then $P(B|A) = P(B)$

or $P(A|B) = P(A)$

Events involving And

① If A, B are two events

then $P(A \text{ and } B) = P(A) \cdot P(B|A)$

② If A, B are independent, then

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Baye's thm

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

$$P(A \text{ and } B) = P(B) \cdot P(A|B)$$

$$\frac{P(A|B)\, P(B)}{P(B)} = \frac{P(A) \cdot P(B|A)}{P(B)}$$

$$\boxed{P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}}$$

Exercises

1. $S = \{1, 2, 3, 4, \ldots, 15\}$

a single number is to be selected at random

given

A: the selected # is even

B: the selected # is a multiple of 4

C: the selected # is a prime #

find each probability

$A = \{2, 4, 6, 8, 10, 12, 14\}$

$B = \{4, 8, 12\}$

$C = \{2, 3, 5, 7, 11, 13\}$

$A \text{ and } B = \{4, 8, 12\}$

$A \text{ and } C = \{2\}$

$B \text{ and } C = \{\}$

(a) $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{7}{15}$

(b) $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{15} = \dfrac{1}{5}$

(c) $P(C) = \dfrac{n(C)}{n(S)} = \dfrac{6}{15} = \dfrac{2}{5}$

(d) $P(A \text{ and } B) = \dfrac{n(A \text{ and } B)}{n(S)} = \dfrac{3}{15} = \dfrac{1}{5}$

(e) $P(A \text{ and } C) = \dfrac{n(A \text{ and } C)}{n(S)} = \dfrac{1}{15}$

(f) $P(B \text{ and } C) = \dfrac{n(B \text{ and } C)}{n(S)} = \dfrac{0}{15} = 0$

(g) $P(A|B) = \dfrac{P(A \text{ and } B)}{P(B)} = \dfrac{n(A \text{ and } B)}{n(B)}$

$= \dfrac{3}{3} = 1$

(h) $P(B|A) = \dfrac{P(A \text{ and } B)}{P(A)} = \dfrac{n(A \text{ and } B)}{n(A)}$

$= \dfrac{3}{7} =$

(i) $P(C|A) = \dfrac{P(A \text{ and } C)}{P(A)} = \dfrac{1}{7}$

(j) $P(A|C) = \dfrac{P(A \text{ and } C)}{P(C)} = \dfrac{1}{6}$

Exercise 2

Given a family with 3 children find the probability of ~~boy birth and girl birth~~ each event. Assume probabilities of boy birth & girl birth are both 1/2

(a) All are girls

$$P(g_1 \cap g_2 \cap g_3) = P(g_1) \cdot P(g_2 | g_1) \cdot P(g_3 | g_1 \cap g_2)$$
$$=$$

(b) all are boys

(c) The oldest two are boys given that there are at least two boys

Reminders

11.2, 11.3 due 04/08

Exam #3 04/26

11.5 Expected Value & Simulation

Random Variable

Expected Value

Given a random variable $X = x_1, x_2, \ldots, x_n$ and corresponding probabilities of these values occurring are $P(x_1), P(x_2), \ldots, P(x_n)$

Then the expected value of X is

$$E(X) = \sum_{i=1}^{n} x_i P(x_i)$$

Here each random variable represent a particular event

Example

1. Find the expected # of boys for a 3-child family (assume boys & girls are equally likely)

$S = \{ ggg, ggb, gbg, bgg, gbb, bgb, bbg, bbb \}$

x # of boys	$P(x)$	$x \cdot P(x)$
0	1/8	$0 \cdot 1/8 = 0$
1	3/8	$1 \cdot 3/8 = 3/8$
2	3/8	$2 \cdot 3/8 = 6/8$
3	1/8	$3 \cdot \frac{1}{8} = 3/8$

$$E(x) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 3/2$$

Expected # of boys

Example

2. Finding expected winnings

A player pays $3 to play the following game

He tosses three fair coins and receives

back payoffs of $1 if he tosses no heads,

$2 for one head

$3 for 2 heads

$4 for 3 heads

Find the player's expected net winnings for this game

$S = \{ttt, htt, tht, tth, hht, hth, thh, hhh\}$

# of heads	Pay off	Net winning X	P(x)	x . P(x)
0	$1	-$2	1/8	-$ 2/8
1	$2	-$1	3/8	-$ 3/8
2	$3	0	3/8	0
3	$4	$1	1/8	$ 1/8

$$E(x) = -\frac{2}{8} - \frac{3}{8} + \frac{1}{8} = -\frac{4}{8} = -\frac{1}{2}$$

-$0.50

What is the fair cost to play this game (04)

The game cost \$0.50 too high
so a fair cost will be $3 - 0.5 = \$2.50$

x	$P(x)$	$x \cdot P(x)$
$-\frac{3}{2}$	$\frac{1}{8}$	$\frac{-3}{16}$
$-\frac{1}{2}$	$3/8$	$-\frac{3}{16}$
$\frac{1}{2}$	$3/8$	$\frac{3}{16}$
$\frac{3}{2}$	$\frac{1}{8}$	$\frac{3}{16}$
		0

Expected Net winning of a fair game $= \$0$

(start here).

OS

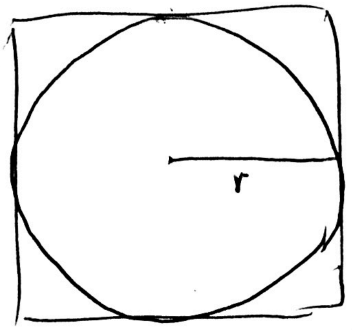
Simulation

Simulation method are called monte-carlo method

show code to estimate pi using monte carlo

Area of circle = πr^2

Area of square = $(2r)^2 = 4r^2$



$$\frac{\text{Area of circle}}{\text{Area of square}} = \frac{\pi \cancel{r^2}}{4 \cancel{r^2}} = \frac{\pi}{4}$$

~~Area of circle~~

$$\pi = 4 \cdot \frac{\text{Area of circle}}{\text{Area of square}}$$

$$= 4 \cdot \frac{\text{\# of pts in circle}}{\text{\# of pts in square}}$$

01

11.5 Exercises

1. Five thousand raffle tickets are sold. One first prize of \$1000, two second prizes of \$500 each, and three third prizes of \$100 each will be awarded (all winners selected randomly)

(a) If you purchased one ticket what are your expected gross winning

	$P(x)$	x gross winning	$x \cdot P(x)$
no prize	$4994/5000$	0	0
1st prize	$1/5000$	1000	$\frac{1000}{5000} = \frac{10}{50}$
2nd prize	$2/5000$	500	$\frac{1000}{5000} = \frac{10}{50}$
3rd prize	$3/5000$	100	$\frac{300}{5000} = \frac{3}{50}$

$$E(X) = \frac{10}{50} + \frac{10}{50} + \frac{3}{50} = \frac{23}{50} = \$0.46$$

(b) If you purchased ten tickets, what are your expected gross winning

$$\left(\frac{10}{5000} \cdot 1000\right) + \left(\frac{20}{5000} \cdot 500\right) + \left(\frac{30}{5000} \cdot 100\right)$$

$$= \frac{23}{5} = \$4.6$$

(b) If you purchased 10 tickets, what are your expected gross winning?

$$E(X) = \left(\frac{10}{5000} \cdot 1000\right) + \left(\frac{20}{5000} \cdot 500\right) + \left(\frac{30}{5000} \cdot 100\right)$$

$$\$ \frac{10}{5} + \frac{10}{5} + \frac{3}{5} = \$ \frac{23}{5} = \$4.60$$

(c) If the tickets were sold for $1 each, how much profit goes to the raffle sponsor

Prizes	$n(x)$ number of tickets	Cost	Payoff	x Profit	$x \cdot n(x)$
0	4994	$1	0	$ 1	$ 4994
1st	1	$1	$ 1000	- $999	- $ 999
2nd	2	$1	$ 500	- $499	- $ 998
3rd	3	$1	$ 100	- $ 99	- $ 297
					$ 2700

Total profit =

Simulation

A Coin was actually tossed 100 times, producing the ffg sequence of outcomes. Read from left to right across the top row, then left to right across the second row and so on,

Using the sequence, find the empirical probability of each of the ffg. Round to 3 d p

(a) two consecutive heads $\frac{48}{199} \approx 0.241$

(b) two consecutive tails $\frac{49}{199} \approx 0.246$

(c) 3 consecutive tosses of the same outcome $\frac{48}{198} \approx 0.242$

Simulation Exercise

h h t h t	t h t h h	t t t t t	t t h t t	h t h h h
t h t t h	h h h h h	h t t h h	h t h h t	h h t t h
t h t h t	h h t t t	h h h h h	t t h t h	t h t h h
t h h h h	h h h h t	t h t t h	t h h t h	t h h t t
t h h t t	t h t t t	h t h h t	t h h h t	h t t t t
h t t h h	h t t t t	t t t h t	t t t t h	t h t h h
h h h h h	t t h h t	t t h h t	h t h t t	h h h h t
h t h t t	h t t t t	h h t t h	t t t h h	t h t t h