Final Exam - Wed Dec 9, (7:30 am - 9:30 am)

Thursday, November 19, 2020 Send out exam 5 (Saturday 11/21)

$$\int f(x)\,dx = F(x) \qquad (F' = f)$$

5.5 Substitution Rule

The Fundamental theorem tells us to find antiderivatives in order to evaluate definite and indefinite integrals.

To find antiderivatives, we look up our table of antiderivatives (see 5.4)

However, if the table of antiderivatives does not tell us $F(x)$ (antiderivative) of $f(x)$

Say we do not want to use substitution, we can use integration by part — see Calc 2

Example

$$\int 2x\sqrt{1+x^2}\,dx$$

(we do not have an antiderivative for $2x\sqrt{1+x^2}$ in the table of antiderivatives (5.4))

Set $u = 1+x^2$

$\frac{du}{dx} = 2x \qquad \left(du = 2x \cdot dx \ , \ \frac{du}{2x} = dx \right)$

$$\int 2x\sqrt{1+x^2}\,dx = \int \cancel{2x}\cdot\sqrt{u}\cdot\frac{du}{\cancel{2x}} = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

Substitution Rule (indefinite Integrals)

If $u = g(x)$ is a differentiable function

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$\int 2x\sqrt{1+x^2}\,dx$ $\quad f(x) = \sqrt{x}$, $g(x) = 1+x^2$

$f(g(x)) = \sqrt{1+x^2}$

$g'(x) = 2x$

Example

1. Find $\int x^3\cos(x^4+2)\,dx$ $\qquad \boxed{u = x^4+2}$

$\frac{du}{dx} = 4x^3 \qquad du = 4x^3\cdot dx \qquad dx = \frac{du}{4x^3}$

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$$\int x^3 \cdot \cos(x^4+2)\, dx = \int \cancel{x^3} \cdot \cos(u) \cdot \frac{du}{4\cancel{x^3}} = \frac{1}{4}\int \cos(u)\, du = \frac{1}{4}\sin(u) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

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(2) Evaluate $\int \sqrt{2x+1}\; dx \qquad u = 2x+1$

$u = 2x+1$

$\frac{du}{dx} = 2, \quad du = 2\,dx, \quad dx = \frac{du}{2}$

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2} \cdot \frac{u^{1/2+1}}{\frac{1}{2}+1} + C = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

(3) find $\int \frac{x}{\sqrt{1-4x^2}}\, dx \qquad u = 1-4x^2$

$u = 1-4x^2$

$\frac{du}{dx} = -8x, \quad du = -8x\, dx, \quad dx = \frac{du}{-8x}$

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-8\cancel{x}} = \int \frac{du}{-8\sqrt{u}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\int \frac{du}{u^{1/2}} = -\frac{1}{8}\int u^{-\frac{1}{2}}\, du$$

$$= -\frac{1}{8}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C = -\frac{1}{8}\cdot\frac{2}{1}u^{1/2} + C = -\frac{2}{8}u^{1/2} + C = -\frac{1}{4}\sqrt{u} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Substitution for definite integral

If g' continuous on $[a,b]$, f continuous on range of $\underline{u = g(x)}$

then
$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Example

$\int_0^4 \sqrt{2x+1}\; dx \qquad u = 2x+1, \quad \frac{du}{dx} = 2, \quad du = 2\,dx$

$dx = \frac{du}{2}$

$u = 2x+1$

$a=0, \quad b=4$

$g(a) = 2\cdot a + 1 \quad , \quad g(0) = 2\cdot 0 + 1 = 1$

$g(b) = 2\cdot b + 1 \quad , \quad g(4) = 2\cdot 4 + 1 = 9$

$$\int_0^4 \sqrt{2x+1}\, dx = \int_1^9 \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2}\int_1^9 \sqrt{u}\, du$$

use FTC $= \frac{1}{2}\left[\frac{2}{3} u^{3/2} \Big|_{u=1}^{u=9} \right]$

$$= \frac{1}{2}\left[\frac{2}{3}\left(9^{3/2}\right) - \frac{2}{3} 1^{3/2} \right]$$

$$= \frac{1}{2}\left[\frac{2}{3}\cdot 27 - \frac{2}{3} \right]$$

$$= \frac{1}{2}\left[\frac{54}{3} - \frac{2}{3} \right]$$

$$= \frac{1}{2}\cdot\frac{52}{3} = \frac{26}{3}$$

Aside

$9^{3/2} = \left(9^{1/2}\right)^3$

$= 3^3 = 27$

Aside

$\int_a^b f(x)\,dx = F(b) - F(a)$

where $F' = f$

F is antiderivative of f

Find antiderivative of $\sqrt{u} = u^{1/2}$

$\frac{d}{dx}(?) = u^{1/2}$

$\frac{d}{dx}\left(\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c \right) = u^{1/2}$

$\frac{d}{dx}\left(\frac{2}{3} u^{3/2} + c \right) = u^{1/2}$

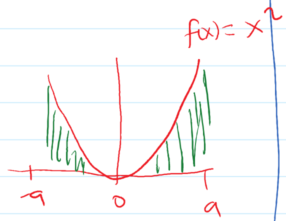
Integral of Symmetric functions

Suppose f is continuous on $[-a, a]$

(1) If f is even $\left(f(-x) = f(x) \right)$

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$$\int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$$



(2) If f is odd $\left(f(-x) = -f(x) \right)$

$$\int_{-a}^{a} f(x)\,dx = 0$$

example of symmetric functions

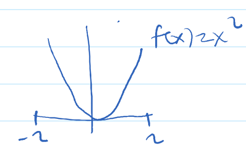
(1) Even function (symmetric about y-axis)

$f(-x) = f(x)$ (reflexive)

example $f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

$= f(x)$



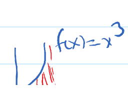
(2) odd functions (symmetric about origin) (rotational)

$f(-x) = -f(x)$

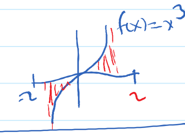
example $f(x) = x^3$

$f(-x) = (-x)^3 = -x \cdot -x \cdot -x$

$= -x^3$



$= -f(x)$



$= -x^3$

$= -f(x)$

1. HW 5.2, 5.3, 5.4, 5.5 due Saturday 11/28

2. Exam #5 due Monday 11/30

3. Final Exam on Wed 12/09 7:30 am –
(webassign + lockdown browser)
you will have to turn on your camera

(I will make a practise set for the final exam) (will be on webassign later today)

If you recall, we skipped # 4.7 (optimization)
(I will go over 4.7 on Wednesday 11/25 (no HW on 4.7))