Final Exam - Wed Dec 9, (7:30am - 9:30 am)
Send out exam 5 (Saturday 11/21)

$$\int f(x)\,dx = F(x) \qquad (F' = f)$$

5.5 Substitution Rule

The Fundamental theorem tells us to find antiderivatives in order to evaluate definite and indefinite integrals.

To find antiderivatives, we look up our table of antiderivatives (see 5.4)

However, if the table of antiderivatives does not tell us $F(x)$ (antiderivative) of $f(x)$

Say we do not want to use substitution, we can us integration by part
see Calc 2

Example

$$\int 2x\sqrt{1+x^2}\,dx$$

(we do not have an antiderivative for $2x\sqrt{1+x^2}$ in the table of antiderivatives (5.4))

Set $u = 1+x^2$

$$\frac{du}{dx} = 2x \qquad \left(du = 2x \cdot dx \;, \quad \frac{du}{2x} = dx \right)$$

$$\int 2x\sqrt{1+x^2}\,dx = \int \cancel{2x}\cdot\sqrt{u}\cdot\frac{du}{\cancel{2x}} = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

Substitution Rule

If $u = g(x)$ is a differentiable function

$$\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$$

$f(x) = \sqrt{x}$
$g(x) = 1+x^2$

$$\int 2x\sqrt{1+x^2}\,dx$$

$f(g(x)) = \sqrt{1+x^2}$
$g'(x) = 2x$

Example

1. Find $\int x^3\cos(x^4+2)\,dx$ $\quad \boxed{u = x^4+2}$

$$\frac{du}{dx} = 4x^3 \qquad du = 4x^3\cdot dx \qquad dx = \frac{du}{4x^3}$$

$\frac{du}{dx} = 4x^3 \qquad du = 4x^3 \cdot dx \qquad dx = \frac{du}{4x^3}$

$$\int x^3 \cdot \cos(x^4+2)\,dx = \int \cancel{x^3} \cdot \cos(u) \cdot \frac{du}{4\cancel{x^3}} = \frac{1}{4}\int \cos(u)\,du = \frac{1}{4}\sin(u) + C$$
$$= \frac{1}{4}\sin(x^4+2) + C$$

---

(2) Evaluate $\int \sqrt{2x+1}\,dx \qquad u = 2x+1$

$u = 2x+1$

$\frac{du}{dx} = 2, \quad du = 2\,dx, \quad dx = \frac{du}{2}$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \frac{du}{2} = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2} \cdot \frac{u^{1/2+1}}{\frac{1}{2}+1} + C = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$
$$= \frac{1}{3}u^{3/2} + C$$
$$= \frac{1}{3}(2x+1)^{3/2} + C$$

(3) find $\int \frac{x}{\sqrt{1-4x^2}}\,dx \qquad u = 1-4x^2$

$u = 1-4x^2$

$\frac{du}{dx} = -8x, \quad du = -8x\,dx, \quad dx = \frac{du}{-8x}$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{-8\cancel{x}} = \int \frac{du}{-8\sqrt{u}} = -\frac{1}{8}\int \frac{du}{\sqrt{u}} = -\frac{1}{8}\int \frac{du}{u^{1/2}} = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$$
$$= -\frac{1}{8}\,\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = -\frac{1}{8}\,\frac{u^{1/2}}{1/2} + C = -\frac{1}{8}\cdot\frac{2}{1}u^{1/2} + C = -\frac{2}{8}u^{1/2} + C = -\frac{1}{4}\sqrt{u} + C$$
$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Substitution for definite integral

if g' continuous on $[a,b]$, f continuous on range of $u = g(x)$

then $$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$