5.4 Indefinite Integral and the Net change Theorem

First I will distinguish between

$$\int_a^b f(x)\,dx \quad \text{vs} \quad \int f(x)\,dx \quad \text{where} \quad \int f(x)\,dx = F(x)$$

5.1
5.2 - Definite integral
5.3
- area under the curve $y=f(x)$
- It is a number

- Indefinite integral
- It is a function

and $F'(x) = f(x)$

$$\frac{d}{dx}\Big(\underbrace{\frac{x^3}{3}+c}_{\text{antiderivative}}\Big) = \underbrace{x^2}_{\text{derivative}}$$

(more elegant) $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$

Table of indefinite integrals

(Table of antiderivatives)

⓪ $\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

① $\int c\,f(x)\,dx = c\int f(x)\,dx$

② $\int k\,dx = kx + c$

③ $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c \quad (n \neq -1)$

④ $\int e^x\,dx = e^x + c$

⑤ $\int \sin(x)\,dx = -\cos(x) + c$

⑥ $\int \sec^2(x)\,dx = \tan(x) + C$

⑦ $\int \sec(x)\tan(x)\,dx = \sec(x) + c$

⑧ $\int \frac{1}{\ldots}\,dx = \tan^{-1}(x) + c$

Recall
$\frac{d}{\ldots}\left(\tan^{-1}(x)\right) = \frac{1}{\ldots}$

✓

8) $\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + c$

Recall

$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{x^2+1}$

9) $\int \sinh(x) = \cosh(x) + C$

10) $\int \frac{1}{x}\,dx = \ln|x| + c$

11) $\int$

12) $\int \cos(x)\,dx = \sin(x) + C$

13) $\int \csc(x)\,dx = -\cot(x) + c$

14) $\int \csc(x)\cot(x)\,dx = -\csc(x) + C$

15) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$

16) $\int \cosh(x)\,dx = \sinh(x) + c$

Exercises

Find indefinite integral

1) $\int (10x^4 - 2\sec^2 x)\,dx = \int 10x^4\,dx - \int 2\sec^2 x\,dx$

$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$

$= 10 \cdot \frac{x^5}{5} - 2\tan x + C$

$= 2x^5 - 2\tan x + C$

Aside

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$

$\int \sec^2 x\,dx = \tan x + C$

$10 \cdot \frac{x^5}{5} + C_1 - 2\tan x + C_2$

$2x^5 - 2\tan x + (C_1 + C_2)$

$2x^5 - 2\tan x + C$

2) $\int_0^3 (x^3 - 6x)\,dx = \boxed{\int_0^3 x^3\,dx} - \boxed{\int_0^3 6x\,dx}$

Aside

FTC2

$$= \left|\int_0 x \, dx\right| - \left|\int_0 6x \, dx\right|$$

use FTC 2

$$= \frac{x^4}{4}\Big|_{x=0}^{x=3} - 6\,\frac{x^2}{2}\Big|_{x=0}^{x=3}$$

$$= \left(\frac{3^4}{4} - \frac{0^4}{4}\right) - \left(3\cdot 3^2 - 3\cdot 0^2\right)$$

$$= \frac{81}{4} - 27 = \underline{-6.75}$$

Aside

FTC 2

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$F' = f$

F (antiderivative)

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

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(3) $$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \int \frac{1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta = \int \csc\theta\cdot\cot\theta\,d\theta$$

$$= -\csc\theta + C$$

Net change Theorem (application of FTC 2)

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

FTC 2

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

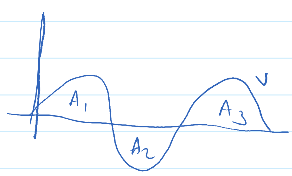
where $F' = f$

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example

If $s(t)$ = displacement,

and $v(t)$ = velocity



$$\int_{t_1}^{t_2} v(t)\,dt = A_1 - A_2 + A_3 \quad \boxed{\text{displacement}}$$

$$\int_{t_1}^{t_2} |v(t)|\,dt = A_1 + A_2 + A_3 \quad \boxed{\text{total distance travelled}}$$

$v(t) = s'(t)$

$$\boxed{\int_{t_1}^{t_2} v(t)\,dt = s(t_2) - s(t_1)}$$

(displacement)

$$\int_{t_1}^{t_2} |v(t)|\,dt = \underline{\text{total distance travelled}}$$

Similarly $a(t) = v'(t)$

So $$\int_{t_1}^{t_2} a(t)\,dt = v(t_2) - v(t_1)$$

Example

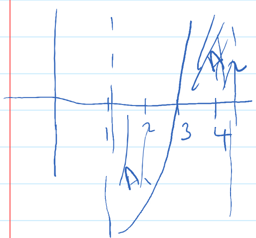
A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$

(a) Find the displacement of the particle during the time period $1 \le t \le 4$

$t_1 = 1 , \quad t_2 = 4$

(b) Find the distance travelled during this time period.

$$\int_1^4 v(t)\,dt = -A_1 + A_2$$



$v(t) \le 0$ on $[1,3]$ and $v(t) \ge 0$ on $[3,4]$

$$\int_1^4 |v(t)|\,dt = A_1 + A_2$$

(a) $$\int_{t_1=1}^{t_2=4} v(t)\,dt = \int_1^4 t^2 - t - 6 = \int_1^4 t^2\,dt - \int_1^4 t\,dt - \int_1^4 6\,dt$$

$$= \frac{t^3}{3}\Big|_{t=1}^{t=4} - \frac{t^2}{2}\Big|_{t=1}^{t=4} - 6t\Big|_{t=1}^{t=4}$$

$$= \left(\frac{4^3}{3} - \frac{1^3}{3}\right) - \left(\frac{4^2}{2} - \frac{1^2}{2}\right) - \left(6(4) - 6(1)\right)$$

$$= \left(\frac{64}{3} - \frac{1}{3}\right) - \left(\frac{16}{2} - \frac{1}{2}\right) - \left(24 - 6\right)$$

$$= \frac{63}{3} - \frac{15}{2} - 18 = -\frac{9}{2}$$

(b) find total distance

$$\int_1^4 |v(t)|\,dt = \int_1^3 -(t^2 - t - 6)\,dt + \int_3^4 t^2 - t - 6\,dt$$

$$\approx 10.17$$