5.3 The fundamental Theorem of Calculus

Part 1 (FTC 1)

If f is continuous on $[a, b]$

then the function g defined by

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b)

and $\boxed{g'(x) = f(x)}$

Aside on FTC 1

$$g(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx}(g(x)) = \frac{d}{dx}\int_a^x f(t)\,dt$$

$$g'(x) = \frac{d}{dx}\int_a^x f(t)\,dt$$

$$\boxed{f(x) = \frac{d}{dx}\int_a^x f(t)\,dt}$$

Fundamental Theorem of calculus part 2 (FTC 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is the antiderivative of f

that is $F' = f$

Explain FTC 1

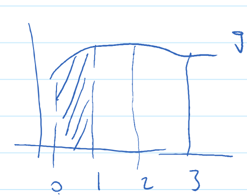
f is continuous on $[a, b]$

$$g(x) = \int_a^x f(t)\,dt \quad \text{on} \quad a \le x \le b$$

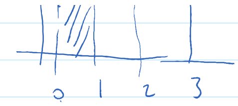
Example

$0 \le x \le 3$, $f(x)$ is continuous on $[0, 3]$

$$g(x) = \int_0^x f(t)\,dt \quad \text{on} \quad 0 \le x \le 3$$



$$g(x) = \int_0 f(t)\,dt \quad \text{on} \quad 0 \le x \le 3$$



$$g(1) = \int_0^1 f(t)\,dt$$

$$g(2) = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt$$

$$= g(1) + \int_1^2 f(t)\,dt$$

$$g(3) = \int_0^2 f(t)\,dt + \int_2^3 f(t)\,dt$$

$$= g(2) + \int_2^3 f(t)\,dt$$

Example use FTC I to find the derivative of $\int_0^x \sqrt{1+t^2}\,dt$

FTC I

$$f(x) = \frac{d}{dx}\int_a^x f(t)\,dt$$

$$f = F' = \frac{d}{dx}\left(F\right)$$

we say F is antiderivative of f

$$F' = f$$

Correspondingly,

$$g(x) = \int_a^x f(t)\,dt$$

$$g'(x) = f(x)$$

g is antiderivative of f

$$\sqrt{1+x^2} = \frac{d}{dx}\int_0^x \sqrt{1+t^2}\,dt$$

$$f = F' = \frac{d}{dx}(F)$$

FTC I (existence of antiderivatives)

let f be continuous on $[a, b]$

Aside

$$g(x) = \int_a^x f(t)\,dt$$

let f be continuous on $[a, b]$

then

$$g(x) = \int_a^x f(t)\,dt \quad \text{on} \quad a \le x \le b$$

is continuous on $[a,b]$, differentiable on (a,b)

and $g'(x) = f(x)$

$$g(x) = \int_a^x f(t)\,dt$$

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_a^x f(t)\,dt$$

$$g'(x) = \frac{d}{dx} \int_a^x f(t)\,dt$$

$$\boxed{f(x) = \frac{d}{dx} \int_a^x f(t)\,dt}$$

Example

Find $\frac{d}{dx} \int_1^{x^4} \sec t \, dt$

Set $x^4 = u \qquad \frac{du}{dx} = 4 \cdot x^3$

$$\frac{d}{dx} \int_1^{x^4} \sec t \, dt = \frac{d}{dx} \int_1^{u} \sec t \, dt$$

$$= \left[\frac{d}{du} \int_1^u \sec t \, dt \right] \cdot \frac{du}{dx}$$

$$= \sec u \cdot 4x^3$$

$$= 4x^3 \cdot \sec x^4$$

FTC 2

If f is continuous on $[a, b]$

then $\int_a^b f(t)\,dt = F(b) - F(a)$ where $F' = f$ (this means that F is an antiderivative of f)

Aside

$\frac{d}{dx}(F) = F' = f$ $\qquad$ F is antiderivative of f

$\frac{d}{dx}\left[\underbrace{\int_a^x f(t)\,dt}_{=g}\right] = f$ $\qquad$ so in FTC1, $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f

use FTC 2 to solve $\int_1^2 x^2\,dx$

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where} \quad F' = f$$

$$\frac{d}{dx}(F) = F' = f$$

Aside

$f = x^n$, $F = \frac{x^{n+1}}{n+1} + C$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + c\right) = x^n$$

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\Bigg|_{x=1}^{x=2}$$

$$\frac{d}{dx}\left(\frac{x^{2+1}}{2+1}\right) = x^2$$

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$$

so $f = x^2$, $F = \frac{x^3}{3}$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{5}{3}$$

$$\frac{d}{dx}\Big(\underbrace{\frac{x^3}{3} + c}_{\text{antiderivative}}\Big) = \underbrace{x^2}_{\text{derivative}}$$

$$\frac{d}{dx}(f(x)) = \underbrace{f'}_{\text{derivative of } f}$$

F, set $F' = f$

$$\frac{d}{dx}\Big(\underbrace{F}_{\text{antiderivative of } f}\Big) = F' = f$$

Example 2

Evaluate $\int_3^6 \frac{dx}{x}$

$$\int_3^6 \frac{dx}{x} = \ln(x)\Bigg|_{x=3}^{x=6}$$

$$= \ln(6) - \ln(3)$$

$$= \ln\left(\frac{6}{3}\right)$$

aside

$$\frac{d}{dx}(\ln(x) + c) = \frac{1}{x}$$

Recall

$$\log a - \log b = \log \frac{a}{b}$$

$$\ln a - \ln b = \ln\left(\frac{a}{b}\right)$$

FTC 2

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F' = f$

$$\frac{d}{dx}(F) = F' = f$$

F is antiderivative of f

$= \ln(2)$

example 3 $\displaystyle\int_0^b \cos(x)\,dx = \sin(x)\Big|_{x=0}^{x=b}$

$= \sin(b) - \sin(0)$

$= \sin(b)$

Aside

$\frac{d}{dx}(\,?\,) = \cos(x)$

$\frac{d}{dx}(\sin(x)) = \cos(x)$

FTC 2

$\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

where $F' = f$