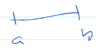
Thursday, November 12, 2020 8:18 AM

5.2 Definite Integral



$\Delta x = \frac{b-a}{n}$

In 5.1 recall

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot \Delta x = \lim_{n\to\infty}\left[f(x_1^*)\cdot\Delta x + f(x_2^*)\cdot\Delta x + \cdots + f(x_n^*)\cdot \Delta x\right]$$

where Δx is the size of the partition

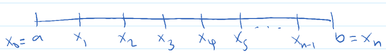
for example if f is defined on domain $[a, b]$

$$\Delta x = \frac{b-a}{n}$$

Definite Integration

If f is a function defined on $[a,b]$ and $[a,b]$ is divided into n subintervals of equal width $\Delta x = \frac{b-a}{n}$

let $x_0 = a, x_1, x_2, \ldots, x_n = b$ be the endpoints of these subintervals



and we let $x_1^*, x_2^*, \ldots, x_n^*$ be sample points in these subintervals

So x_i^* lies the ith subinterval $[x_{i-1}, x_i]$

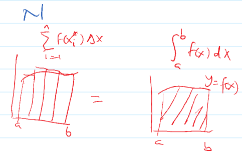
Then
$$\underbrace{\int_a^b f(x)\,dx}_{\text{continuous}} = \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{(Discrete)}}$$
(provided the limit exist)

$\boxed{\text{If this limit exists, we say } f \text{ is integrable on } [a,b]}$

we can write this limit using the famous $\varepsilon - N$ definition

for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(x_i^*)\,\Delta x \right| < \varepsilon$$



for every $n > N$ and for every choice x_i^* in $[x_{i-1}, x_i]$

We Note that

1. $\int_a^b f(x)\,dx$ is a number (it does not depend on x)

we can replace with

$$\int_a^b f(x)dx = \int_a^b f(t)\,dt = \int_a^b f(r)\,dr$$

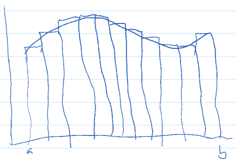
2. The sum $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is called $\boxed{\text{Riemann Sum}}$

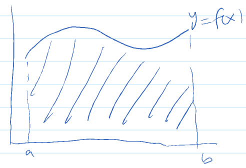
(Definition above says definite integrals can be approximated by Riemann Sum)

If $f \geq 0$, then $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ is sum of areas of the approximating rectangles

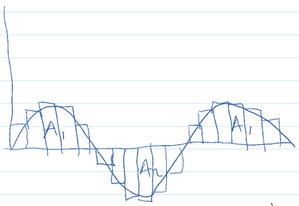
If $f \geq 0$, then $\sum_{i=1} f(x_i)\Delta x$ ...
rectangles

If $f \geq 0$, then $\int_a^b f(x)\,dx$ is the area under the curve $y=f(x)$ from a to b





If f is both positive & negative, Riemann sum is the sum of the areas of rectangles above the x-axis $\boxed{\text{minus}}$ areas of rectangles below the x-axis



$$\int_a^b f(x)\,dx = A_1 - A_2 \quad \text{(net Area)}$$

Exercise

Evaluate the Riemann sum for

$f(x) = x^3 - 6x$, $0 \leq x \leq 3$ with $n = 6$

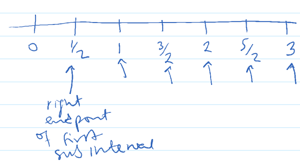
(x_i^* = right end points)

$\uparrow$ sample points

x_{i-1} x_i $\leftarrow$ right endpoint

Solution $f(x) = x^3 - 6x$

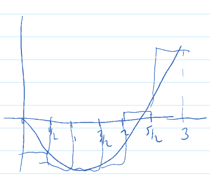
$$\Delta x = \frac{3-0}{6} = \frac{3}{6} = \frac{1}{2}$$



$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x = f(\tfrac{1}{2})\cdot\tfrac{1}{2} + f(1)\cdot\tfrac{1}{2} + f(\tfrac{3}{2})\cdot\tfrac{1}{2} + f(2)\cdot\tfrac{1}{2} + f(\tfrac{5}{2})\cdot\tfrac{1}{2} + f(3)\cdot\tfrac{1}{2}$$

$$= (-2.875)\cdot\tfrac{1}{2} + (-5)\cdot\tfrac{1}{2} + (-5.625)\cdot\tfrac{1}{2} + (-4)\cdot\tfrac{1}{2} + 0.625\cdot\tfrac{1}{2} + 9\cdot\tfrac{1}{2}$$

$$= -3.9375$$



Aside

$f(x) = x^3 - 6x$

$f(\tfrac{1}{2}) = -2.875$

$f(1) = -5$

$f(\tfrac{3}{2}) = -5.625$

$\vdots$

Theorem A

If f is continuous on $[a,b]$ or if f has only a finite number of jump discontinuities, then f is integrable on $[a,b]$

$\int_a^b f(x)\,dx$ exists

Remark

If f is integrable on $[a,b]$ then the choice of x_i^* does no matter

$$\left(x_i^* = x_i \text{ (right endpoints)} \right)$$

x_{i-1} x_i $\quad x_i^* \in [x_{i-1}, x_i]$

Theorem B

continuous

a b

If f is integrable on $[a, b]$, then

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x}$$

Discretize

$a = x_0 \quad x_1 \quad x_2 \quad \cdots \quad x_{n-1} \quad x_n = b$

where $\Delta x = \dfrac{b-a}{n}$, $\quad x_i = a + i\Delta x$

Crash Course on Sums

① $\displaystyle\sum_{i=1}^{n} 1 = \underbrace{1+1+1+\cdots+1}_{n \text{ times}} = n$

② $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

③ $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

④ $\displaystyle\sum_{i=1}^{n} i^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Properties of Sums

(1) $\displaystyle\sum_{i=1}^{n} c a_i = c \cdot \sum_{i=1}^{n} a_i$

② $\displaystyle\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

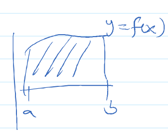
③ $\displaystyle\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Midpoint Rule

In some problem $\left(x_i^* = \text{midpoint } \bar{x}_i \quad x_{i-1} \quad x_i \qquad \bar{x}_i = \dfrac{x_{i-1} + x_i}{2} \right)$

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x \qquad \text{where } \Delta x = \frac{b-a}{n}$$

Properties of definite integral



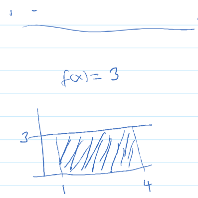
① $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \qquad (a < b)$

$$\boxed{\int_3^1 x^2\,dx = -\int_1^3 x^2\,dx}$$

② $\displaystyle\int_a^a f(x)\,dx = 0 \qquad (a = b)$

$f(x) = 3$

② $\int_a^a f(x)\,dx = 0$ $(a=b)$



③ $\int_a^b c\,dx = c(b-a)$ (c is a constant)

④ $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

⑤ $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

⑥ $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

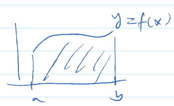
Special Property

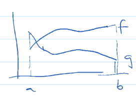
$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

more properties

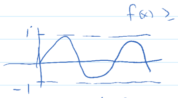
① If $f \geq 0$ for $a \leq x \leq b$

then $\int_a^b f(x)\,dx \geq 0$





$f(x) \geq g(x)$ for $a \leq x \leq b$

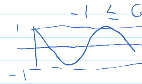


② If $f(x) \geq g(x)$ for $a \leq x \leq b$

then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$-1 \leq \sin(x) \leq 1$

$-1 \leq \cos(x) \leq 1$



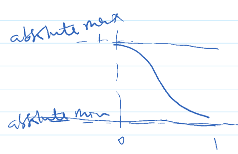
③ If $m \leq f(x) \leq M$ for $a \leq x \leq b$

then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

If f is bounded
(f has an absolute min and absolute max)

example

use prop (3) to estimate $\int_0^1 e^{-x^2}\,dx$



$f(x) = e^{-x^2}$ is decreasing on $[0,1]$

It has absolute maximum at 0

$f(0) = e^{-0} = e^0 = 1$

It has absolute minimum at 1

$f(1) = e^{-1} = \frac{1}{e}$

so we have $\frac{1}{e} \leq e^{-x^2} \leq 1$ on $0 \leq x \leq 1$

So by property 3,

$$\frac{1}{e}(1-0) \leq \int_0^1 e^{-x^2}\,dx \leq 1(1-0)$$

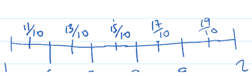
$$\frac{1}{e} \leq \int_0^1 e^{-x^2}\,dx \leq 1$$

Example on Midpoint Rule

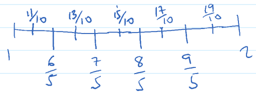
use midpoint rule with $n=5$ to approximate $\int_1^2 \frac{1}{x}\,dx$

$f(x) = \frac{1}{x}$

$\Delta x = \frac{2-1}{5} = \frac{1}{5}$



$$\Delta x = \frac{2-1}{5} = \frac{1}{5} \qquad f(x) = \frac{1}{x}$$



midpoint

$\frac{5}{5}$ $\frac{6}{5}$ $= \frac{1}{2}\left(\frac{5}{5}+\frac{6}{5}\right) = \frac{11}{10}$

$\frac{6}{5}$ $\frac{7}{5}$ $= \frac{1}{2}\left(\frac{6}{5}+\frac{7}{5}\right) = \frac{13}{10}$

$\frac{7}{5}$ $\frac{8}{5}$ $= \frac{1}{2}\left(\frac{7}{5}+\frac{8}{5}\right) = \frac{15}{10}$

$$\int_1^2 \frac{1}{x}\,dx = \frac{1}{5}\left(f\left(\tfrac{11}{10}\right) + f\left(\tfrac{13}{10}\right) + f\left(\tfrac{15}{10}\right) + f\left(\tfrac{17}{10}\right) + f\left(\tfrac{19}{10}\right)\right)$$

$$= \frac{1}{5}\left(\frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19}\right)$$

$=$ (exercise)

Using Thm 3

evaluate $\int_0^3 (x^3 - 6x)\,dx$

$$\left(\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x\right)$$

$f(x) = x^3 - 6x$, $a = 0$, $b = 3$, $\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_i = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$ $\qquad x_i = a + i\,\Delta x$

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\cdot\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{27 i^3}{n^3} - \frac{18 i}{n}\right]\cdot\frac{3}{n}$$

$$= \frac{3}{n}\cdot \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{27 i^3}{n^3} - \frac{18 i}{n}\right)$$

$$= \frac{3}{n} \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{27 i^3}{n^3} - \sum_{i=1}^{n} \frac{18 i}{n}\right]$$

$$= \frac{3}{n} \lim_{n\to\infty} \left[\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{18}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{3}{n}\cdot\frac{27}{n^3}\sum_{i=1}^{n} i^3 - \frac{3}{n}\cdot\frac{18}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{n^4}\,\frac{n^2 (n+1)^2}{2^2} - \frac{54\; n(n+1)}{n^2\cdot 2}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\cdot\frac{(n+1)^2}{n^2} - \frac{54}{2}\cdot\frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\cdot\left(\frac{n+1}{n}\right)^2 - \frac{54}{2}\left(\frac{n+1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\left(\frac{n}{n}+\frac{1}{n}\right)^2 - \frac{54}{2}\left(\frac{n}{n}+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{81}{4}\cdot\left(1+\frac{1}{n}\right)^2 - \frac{54}{2}\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 - \frac{54}{2}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$f(x) = x^3 - 6x$

$$f\left(\frac{3i}{n}\right) = \left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)$$

$$= \frac{3^3 i^3}{n^3} - \frac{18 i}{n}$$

$$= \frac{27 i^3}{n^3} - \frac{18 i}{n}$$

Aside

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\lim_{n\to\infty}\left[\ \cdot\left(1+\frac{}{n}\right) - \frac{}{2}\left(1+\frac{}{n}\right)\right]$$

$$= \frac{81}{4}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 - \frac{54}{2}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)$$

$$= \frac{81}{4}\cdot(1+0)^2 - \frac{54}{2}\cdot(1+0)$$

$$= \frac{81}{4} - \frac{54}{2} = \text{complete yourself}$$