Chapter 5

5.1 The Area and Distance problem

Area under the Curve

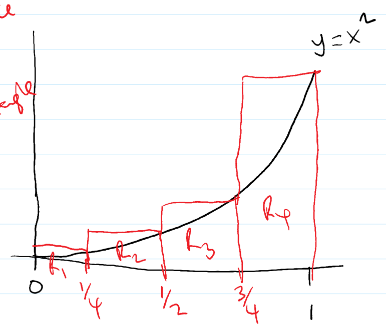
The area of the rectangle will overestimate the area under the curve

we get better estimation by partitioning the rectangles

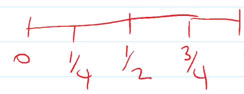
Example

Use rectangles to estimate the area under the parabola $y = x^2$ from 0 to 1

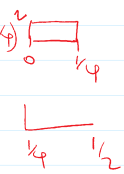
over-estimate by using the right value of each rectangle



Partition



height of the rectangle $(\frac{1}{4})^2$



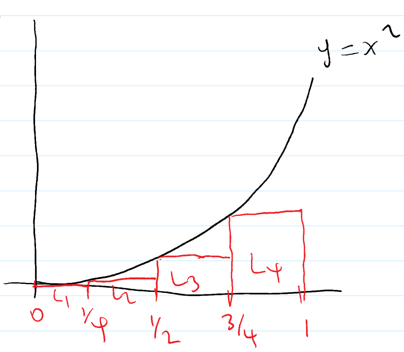
✓

Area under Curve A = Area of R_1 + Area of R_2 + Area of R_3 + Area of R_4

$$= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2$$

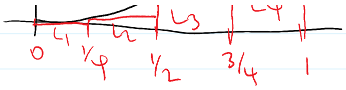
$$= \frac{1}{4}\left(\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}\right)^2 + 1^2\right) = 0.46875$$

underestimate by using the left value of each rectangle



Area = Area of L_1 + Area of L_2 + Area of L_3 + Area of L_4

$$= \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$



$$= \frac{1}{4}\cdot 0^2 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= 0.21875$$

The actual area under the curve lies between

$$\underset{L_n}{0.21875} < \underline{A} < \underset{R_n}{0.46875}$$

if we make the partition smaller $0.21875 \to A$

$0.46875 \to A$

An interesting table in your textbook

n	L_n	R_n
10	0.285000	0.385000
20	0.3087500	0.358750
30	0.3168519	0.3501852
50	0.323400	0.343400
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

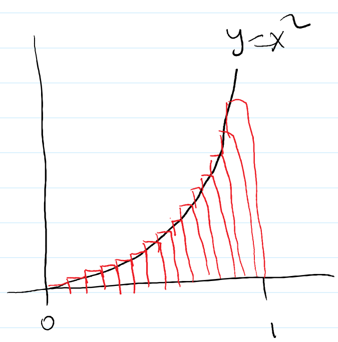
you should notice

$$\lim_{n\to\infty} L_n = \frac{1}{3} \checkmark$$

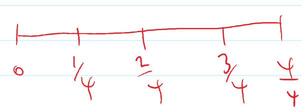
$$\lim_{n\to\infty} R_n = \frac{1}{3} \checkmark$$

Example 2

Show that $\lim_{n\to\infty} R_n = \frac{1}{3}$ in Example 1



Since we want n partitions the length of each partition is $\frac{1}{n}$



$$R_n = \frac{1}{n}\cdot\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{1}{n}\right)\left(\frac{n}{n}\right)^2$$

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n}\cdot\frac{1}{n^2}\left[1^2 + 2^2 + \cdots + n^2\right]$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + \cdots + n^2\right]$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

Aside

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \frac{(n+1)(2n+1)}{6n^2}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{1}{6} \cdot \left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n}{n} + \frac{1}{n}\right)\left(\frac{2n}{n} + \frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

$$= \frac{1}{6}(1+0)(2+0)$$

$$= \frac{2}{6} = \frac{1}{3}$$

we can do similar for $\lim_{n\to\infty} L_n = \frac{1}{3}$ from Example 1

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* Bernhard Riemann

Definition

The Area A that lies under the graph of a continuous function f is the lim of the sum of the areas of approximating rectangles

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots + f(x_n)\cdot\Delta x\right] \quad \text{---} \ (*)$$

If f is increasing
- the rightend points of each partition will overestimate
- the leftend points of each partition will underestimate

(the opposite will happen
if the graph f is decreasing)

we can re-write (*) as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x = \lim_{n\to\infty} \left[f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x \right]$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \, \Delta x}$$

(when we use right end pts)

what if use left end points

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \left[f(x_0) \cdot \Delta x + f(x_1) \cdot \Delta x + \cdots + f(x_{n-1}) \, \Delta x \right] \quad - (**)$$

$$\boxed{A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1}) \, \Delta x}$$

(when we use left end pts)