4.9 Antiderivatives

Definition

A function F is an antiderivative of f on interval I if $F' = f$

$$\frac{d}{dx}(F) = F' = f$$

antiderivative of f

$$\frac{d}{dx}(F) = F' = f$$

find (given)

(F is the antiderivative of f)

Example $f = 2x$ what is the antiderivative of f

derivative

$$\frac{d}{dx}(x^2) = \boxed{2x}$$

antiderivative

So x^2 is an antiderivative of $2x$

observe

$$\frac{d}{dx}(x^2+5) = \boxed{2x}$$

x^2+5 is an antiderivative of $2x$

$$\frac{d}{dx}(x^2-2) = \boxed{2x}$$

x^2-2 is an antiderivative of $2x$

So $\boxed{x^2+C}$ is <u>the</u> most general antiderivative of $2x$ where C is a constant

Theorem

If F is an antiderivative of f on I. Then the most general antiderivative of f on I is

$$F(x) + C \qquad (C \text{ is a constant})$$

Practical application

(given) S = Distance (a function with respect to time t)

find V = derivative of S

find a = Second derivative of S

Here, we are given v, a the antiderivative of v is S

$$\frac{d}{dt}(S) = v$$

Examples

Find the antiderivative of the following functions

(a) $f(x) = \sin(x)$

$$\frac{d}{dx}\left(-\cos(x)\right) = \sin(x)$$

we see that $-\cos(x)$ is an antiderivative of $\sin(x)$

you will agree

$$\frac{d}{dx}\left(-\cos(x) + \pi\right) = \sin(x)$$

The general antiderivative $-\cos(x) + C$ (C is a constant)

Aside

(a) $\frac{d}{dx}(\sin(x)) = \cos(x)$

(b) $\frac{d}{dx}(\cos(x)) = -\sin(x)$

$\frac{d}{dx}\left(-\cos(x)\right)$

$= -\frac{d}{dx}(\cos(x)) = -(-\sin(x))$

$= \sin(x)$

(b) $f(x) = \frac{1}{x}$

$\ln(x)$ is an antiderivative of $\frac{1}{x}$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x} \quad \text{on } (0,\infty)$$

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{x} \quad \text{on } (-\infty,0)$$

The general antiderivative of $\frac{1}{x}$ is $\ln|x| + C$

$$\ln|x| = \begin{cases} \ln(x) & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

example

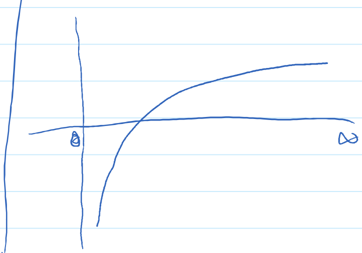
$x = -5$

$\ln(-x) = \ln(-(-5))$

$= \ln(5)$

Aside

$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$



Example

Find the antiderivative of $\frac{3}{x}$

$\frac{d}{dx}\left(3\ln|x| + C\right) = \frac{3}{x}$

$3 \frac{d}{dx}\left(\ln|x| + c\right) = 3\frac{1}{x}$

(c) find the antiderivative of $f(x) = x^n$, $\boxed{n \neq -1}$

$$\frac{d}{dx}\left(? \right) = x^n$$

I am going to cheat (I have done this before)

the antiderivative $\frac{x^{n+1}}{n+1}$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

So $\frac{x^{n+1}}{n+1}$ is an antiderivative of x^n

The general antiderivative of x^n

$$\frac{x^{n+1}}{n+1} + C$$

Aside

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)$$

$$= \frac{1}{n+1} \frac{d}{dx}\left(x^{n+1}\right)$$

$$= \frac{1}{n+1} \cdot (n+1)\, x^{n+1-1}$$

$$= \frac{1}{\cancel{(n+1)}} \cdot \cancel{(n+1)} \cdot x^n$$

$$= x^n$$

Example

find antiderivative of x^2

we know $\boxed{\frac{x^{n+1}}{n+1} + C \text{ is the general antiderivative of } x^n}$

$\frac{x^{2+1}}{2+1} + C$ is the general antiderivative x^2

$$\frac{x^3}{3} + C$$

$\left(F \text{ is an antiderivative of } f \text{ if } f = F' \right)$

Function	Antiderivative
$c\,f(x)$	$c\,F(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
b^x	$\frac{b^x}{\ln(b)}$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
$\cosh(x)$	$\sinh(x)$
$\sinh(x)$	$\cosh(x)$

Aside

① $\frac{d}{dx}(b^x) = b^x \ln(b)$

$$\frac{d}{dx}\left(\frac{b^x}{\ln(b)}\right)$$

$$= \frac{1}{\ln(b)} \cdot \frac{d}{dx}(b^x)$$

$$= \frac{1}{\cancel{\ln(b)}} \cdot b^x \cancel{\ln(b)}$$

$$= b^x$$

Hyperbolic sin $(\sinh(x))$

Hyperbolic cos $(\cosh(x))$

Exercise

Find the antiderivative g such that

$$g'(x) = 4\sin(x) + \frac{2x^5 - \sqrt{x}}{x}$$

$$= 4\sin(x) + \frac{2x^5}{x} - \frac{\sqrt{x}}{x}$$

$\overline{}$ $\overline{}$
x x

$$g' = 4\sin(x) + 2x^4 - \frac{\sqrt{x}}{x}$$

$$4 \cdot \frac{d}{dx}\left(-\cos(x)\right) = 4\sin(x)$$

$$2\,\frac{d}{dx}\left(\frac{x^5}{5}\right) = 2x^4$$

$$\frac{d}{dx}\left(\frac{x^{\frac{1}{2}}}{1/2}\right) = \frac{\sqrt{x}}{x}$$

Aside

$$x^{1/n} = \sqrt[n]{x}$$

$$x^{1/2} = \sqrt{x}$$

$$\frac{x^{\frac{1}{2}}}{\frac{1}{2}} = x^{\frac{1}{2}} \div \frac{1}{2} = x^{\frac{1}{2}} \cdot \frac{2}{1} = \sqrt{x} \cdot 2 = 2\sqrt{x}$$

from the table

using

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$$

$$\frac{d}{dx}\left(\frac{x^{4+1}}{4+1}\right) = x^4$$

$$\frac{\sqrt{x}}{x} = \frac{\sqrt{x}}{x} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{x}{x\sqrt{x}} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) = x^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left(\frac{x^{1/2}}{1/2}\right) = x^{-1/2}$$

The antiderivative

$$g = -4\cos(x) + 2\frac{x^5}{5} - 2\sqrt{x} + C$$