4.5 Curve Sketching (continuation of 4.3)

(we saw how f' helps us sketch f)

(4.3 + some college Algebra + x) ↑ new stuff

Checklist when you want to sketch a function

(A) Domain of the function

example

polynomial functions - Domain is all real numbers $(-\infty, \infty)$

$f(x) = e^x$ $\quad (-\infty, \infty)$

$f(x) = \sqrt{x}$ $\quad (0, \infty)$

$f(x) = \frac{1}{x}$ $\quad (-\infty, 0) \cup (0, \infty)$

$f(x) = \ln(x)$ $\quad (0, \infty)$

$f(x) = \frac{1}{x-2}$ $\quad (-\infty, 2) \cup (2, \infty)$

(B) Intercept

(we find y-intercept by computing $f(0)$)

(we find x-intercept by setting $f(x) = 0$, solving for x)

(x-intercepts or y-intercepts)

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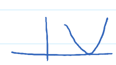
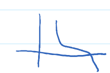
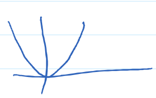
is where the curve either touches  or crosses  the x-axis

$f(x) = x^2$  x-intercept $x = 0$ (multiplicity of 2)

$f(x) = (x-2)^2$  x-intercept $x = 2$ (multiplicity of 2)

(x-intercept is going to touch the x-axis if the multiplicity is even)

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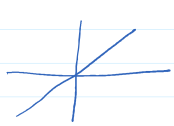
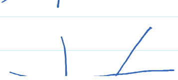
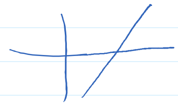
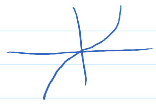
(The number of times the x-intercept occurs)

$f(x) = x$  x-intercept $x = 0$ (multiplicity of 1)

$f(x) = x - 2$  x-intercept $x = 2$ (multiplicity

$f(x) = x-2$  x-intercept $x=2$ (multiplicity of 1)

$f(x) = x^3$  x-intercept $x=0$ (multiplicity of 3)

(c) Symmetry

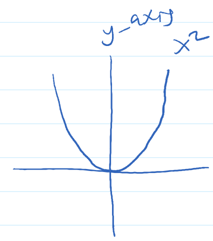
(a) Symmetric about the y-axis (reflexional symmetry) (Even functions)

$$f(-x) = f(x)$$

example

$$f(x) = x^2$$

$$f(-x) = (-x)^2 = (-x)\cdot(-x) = x^2$$



(b) Symmetric about the origin (rotational symmetry) (odd functions)

$$f(-x) = -f(x)$$

Example

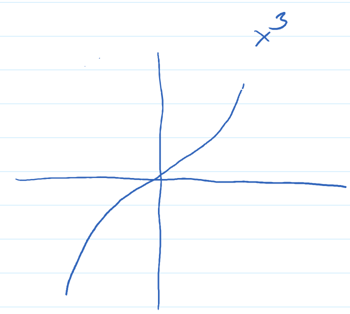
$$f(x) = x^3$$

$$f(-x) = (-x)^3 = (-x)\cdot(-x)\cdot(-x)$$

$$= x^2\cdot(-x)$$

$$= -x^3$$

$$= -f(x)$$



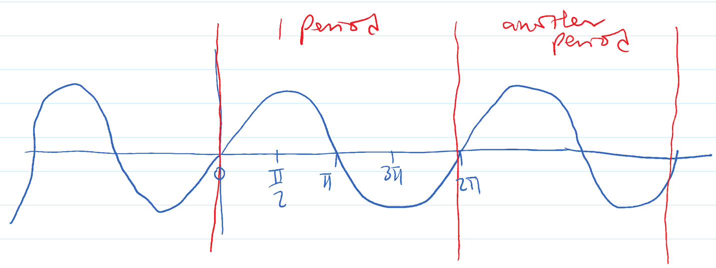
Periodic function

$f(x+p) = f(x)$ for every x in Domain of f

example

$$\sin(x) = \sin(x+2\pi)$$

$$\tan(x) = \tan(x+\pi)$$



(D) Asymptotes

Vertical Asymptotes

$\lim_{x \to \pm a} f(x) = \pm\infty$ then line $x = \pm a$ is a vertical asymptote

Horizontal Asymptotes

$\lim_{x \to \infty} f(x) = L$ then line $y = L$ is an Horizontal asymptote

Slant / oblique Asymptote

Casual definition of horizontal asymptote

$f(x) = \dfrac{P(x)}{q(x)}$, $P(x), q(x)$ are polynomial functions

$P(x) = \boxed{a x^n} + \cdots$ leading coefficient

$q(x) = \boxed{b x^m} + \cdots$ leading coefficient

n = degree of $P(x)$
m = degree of $q(x)$

If $n < m$ then line $y = \frac{a}{b}$ is the horizontal asymptote

If $n > m$ there is no Horizontal asymptote

Example

find the Asymptote of

$$f(x) = \frac{x^3}{x^2+1}$$

$3 > 2$

deg of numerator, deg of denominator

no H.A

we have a slant asymptote

using long division

$$\frac{x^3}{x^2} = x^{3-2} = x^1$$

$$\begin{array}{r} x \\ x^2+1 \overline{\smash{)}\, x^3} \\ -\ (x^3 + x) \\ \hline 0 \quad -x \end{array}$$

$f(x)$

$$\boxed{\frac{x^3}{x^2+1}} = x - \frac{x}{x^2+1}$$

Slant Asymptote

$$\boxed{\text{If } \lim_{x \to \infty}\left(f(x) - x\right) = 0 \text{ then } y = x \text{ is the slant asymptote}}$$

$$f(x) - x = \frac{x^3}{x^2+1} - x$$

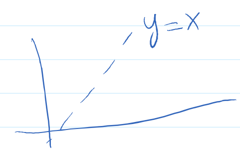
$$= \left(x - \frac{x}{x^2+1}\right) - x = \cancel{x} - \frac{x}{x^2+1} - \cancel{x}$$

$$\lim_{x\to\infty}\left(f(x)-x\right) = \lim_{x\to\infty}\left(\frac{-x}{x^2+1}\right)$$

$$= \lim_{x\to\infty} \frac{-x/x^2}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{-\frac{1}{\infty}}{1+\frac{1}{\infty}}$$

$$= \frac{0}{1+0} = 0$$

$\lim_{x\to\infty}\left(f(x)-x\right) = 0$ so slant asymptote $y = x$



(E) Intervals of increasing or decreasing (Incr/decr test) see 4.3

(F) local maximum and local minimum (see 4.3)

(G) Concavity and point of inflection (see 4.3)

(H) sketch the curve

Aside

long division

$$x \overline{\smash{)}\,x^2} \quad \text{quotient } x; \quad -x^2 \to 0 \qquad \frac{x^2}{x} = x$$

$$\frac{x}{1} + \frac{1}{x} \qquad \frac{x^2+1}{x}$$

$$x \overline{\smash{)}\,x^2+1} \quad \text{quotient } x; \quad -x^2 \to 0+1 \qquad \boxed{\frac{x^2+1}{x} = x + \frac{1}{x}}$$