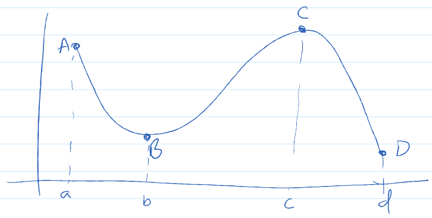
4.3 What does f' tell us about f

f' ← derivative of a function

f ← help us to sketch f



Q can f' find portions of the domain of f where f is increasing or decreasing

(i) $f'(x) > 0$ (slope) in B to C, f is increasing on (b, c)

(ii) $f'(x) < 0$ in A to B, f is decreasing on (a, b)

(iii) $f'(x) < 0$ in C to D, f decreasing on $(c, d$

Increasing / Decreasing Test

(i) If $f'(x) > 0$ on an interval, f is increasing on that interval

(ii) If $f'(x) < 0$ on an interval, f is decreasing on that interval

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Example

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

Find the portion of the domain of f where f is increasing or decreasing

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$f'(x) = 12x(x-2)(x+1)$$

To partition the domain, we need to find the critical points

Set $f'(x) = 0$, solve for x

$12x(x-2)(x+1) = 0$, solve for x

$12x = 0$ or $x - 2 = 0$ or $x + 1 = 0$

$x = 0$ or $x = 2$ or $x = -1$

So the critical points are

$-1, 0, 2$

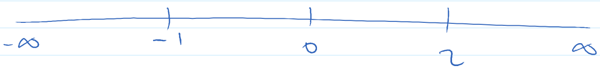
aside

Zeroth product property

If $a.b = 0$ then either $a = 0$, $b = 0$, or both are zero

So the critical points are

$-1, 0, 2$



$a = 0$, $b = 0$, or both are zero

a, b are real numbers

If a, b were not real numbers

Say a, b are from $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$

$2 \cdot 3 \equiv 0 \pmod 6$

Next, we use the increasing/decreasing test

$f'(x) = 12x\,(x-2)\,(x+1)$

x	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-ve$	$-ve$	$-ve$	$-ve$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-ve$	$-ve$	$+ve$	$+ve$	increasing on $(-1, 0)$
$0 < x < 2$	$+ve$	$-ve$	$+ve$	$-ve$	decreasing on $(0, 2)$
$x > 2$	$+ve$	$+ve$	$+ve$	$+ve$	increasing on $(2, \infty)$

Negative $= -ve$

Positive $= +ve$

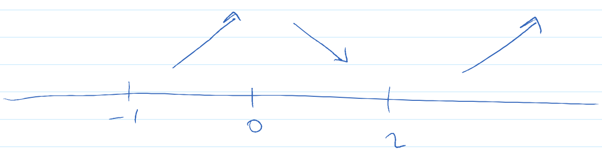
Algebra

$-ve \cdot -ve = +ve$

$+ve \cdot +ve = +ve$

$-ve \cdot +ve = -ve$

$+ve \cdot -ve = -ve$



f has a local max at 0

f has a local min at -1

f has a local min at 2

First derivative Test

Suppose c is a critical point

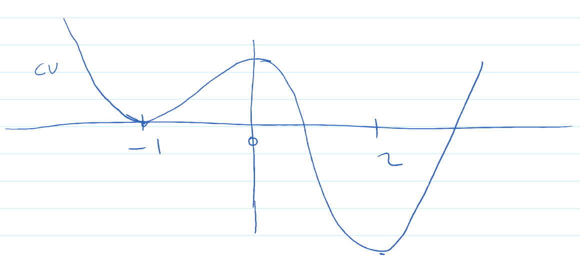
(a) If f' changes from positive to negative at c

f has a local maximum at c

(b) If f' changes from negative to positive at c

f has a local minimum at c

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$



local min at $x = -1$

$f(-1) = 0$

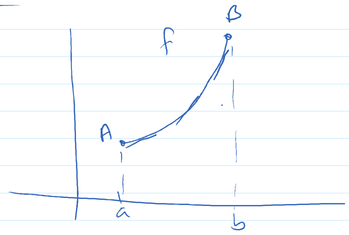
local max at $x = 0$

$f(0) = 5$

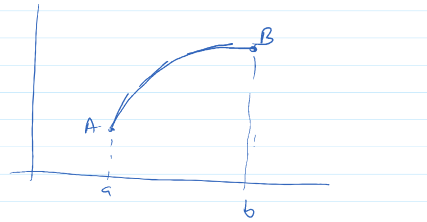
local min at $x = 2$

$f(2) = -27$

Concavity



If a curve lies above tangent
f is called concave upward on (a,b)



If a curve lies below tangent
f is called concave downward on (a,b)

Concavity Test

(a) If $f''(x) > 0$ on an interval, then f concave upward on that interval

(b) If $f''(x) < 0$ on an interval, then f concave downward on that interval

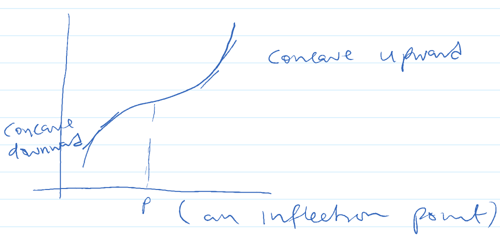
Definition

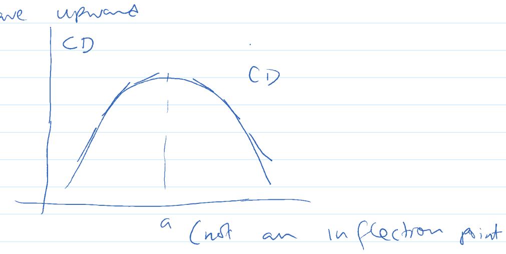
A point P on a curve f is called an inflection point if f is continuous at P and the curve

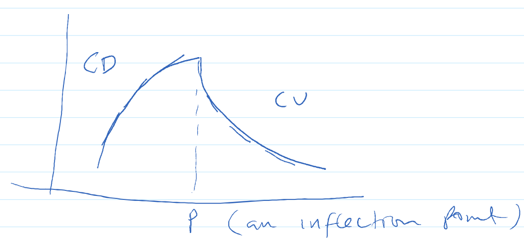
(i) changes from concave upward to concave downward at P

(ii) changes from concave downward to concave upward at P

CD: Concave downward
CV: concave upward







Example

Sketch a possible graph of a function f that satisfies the following:

solution
Increasing/decreasing test

(a) $f'(x) > 0$ on $(-\infty, 1)$ ⇒ f is increasing on $(-\infty, 1)$

$f'(x) < 0$ on $(1, \infty)$ $\Rightarrow$ f is decreasing on $(1, \infty)$

ⓑ $f''(x) > 0$ on $(-\infty, -2)$, $(2, \infty)$
$f''(x) < 0$ on $(-2, 2)$

Concavity test
f is concave upward on $(-\infty, -2)$, $(2, \infty)$
f is concave downward on $(-2, 2)$

Kayode Olumoyin at 11/3/2020 8:14 AM

ⓒ $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$

f has Horizontal asymptote line: $y = -2$

f has Horizontal asymptote line : $y = 0$

Second derivative Test

(converse of the Fermat's theorem + an additional condition)

ⓐ If $f'(c) = 0$, $f''(c) > 0$ then f has a local min at c

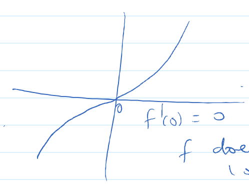
ⓑ If $f'(c) = 0$, $f''(c) < 0$ then f has a local max at c

Recall Fermat's theorem

If f has a local min/max at c then c is a critical point ($f'(c) = 0$)

~~Converse of Fermat's theorem~~
~~Not true~~

(If $f'(c) = 0$, and $f''(c) > 0$ then f has a local min/max at c)



The second derivative test is inconclusive when $f''(c) = 0$

Example

Discuss the curve
$y = x^4 - 4x^3$ with respect to concavity, point of inflection, local max/min

Solution
~~Solution~~
~~$f(x) = x^4 - 4x^3$~~
$f(x) = x^4 - 4x^3$
~~$f'(x) =$~~
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Next find the critical points

Set $f'(x) = 0$, solve for x

$f'(x) = 4x^2(x-3) = 0$

$4x^2(x-3) = 0$

$4x^2 = 0$ or $x - 3 = 0$

$x = 0$ or $x = 3$

aside
zeroth product property

If $a \cdot b = 0$ then either $a = 0$ or $b = 0$ or both are zero

Using 2nd derivative test,

1. $f'(0) = 0$, $f''(0) = 0$ (no information about the critical point 0)

2. $f'(3) = 0$, $f''(3) = 36 > 0$ (there is a local min at 3, $f(3) = -27$)

Using the first derivative test at 0

$f'(x) = 4x^2(x-3)$

x	$4x^2$	$(x-3)$	$f'(x)$	f
$x < 0$	+ve	-ve	-ve	
$0 < x < 3$	+ve	-ve	-ve	

(we do not get a local min/max at 0)

$f''(x) = 12x(x-2) = 0$ $x = 0$ or 2

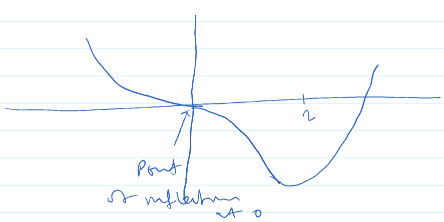
Interval	$12x$	$x-2$	$f''(x)$
$(-\infty, 0)$	-ve	-ve	+ve
$(0, 2)$	+ve	-ve	-ve
$(2, \infty)$	+ve	+ve	+ve

Concavity test

$f''(x) > 0$ on $(-\infty, 0)$, concave upward on $(-\infty, 0)$

$f''(x) < 0$ on $(0, 2)$, concave downward on $(0, 2)$

$f''(x) > 0$ on $(2, \infty)$, concave upward on $(2, \infty)$



Exercise

Sketch the graph

$f(x) = x^{2/3}(6-x)^{1/3}$

Solution

It is easy to show that

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}} \quad , \quad f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

find the critical points

Set $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}} = 0$

we get $x = 4$ (critical point)

Since $f'(0)$ DNE, $x=0$ is also a critical point

$f'(6)$ DNE, $x=6$, is also a critical point

Aside

c is a critical point of f

If $f'(c) = 0$

or

$f'(c)$ DNE

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$

$$f'(0) = \frac{4-0}{0^{1/3}(6-0)^{2/3}} = \frac{4}{0} \quad \text{DNE}$$

$$f'(6) = \frac{4-6}{6^{1/3}(6-6)^{2/3}} = \frac{-2}{0} \quad \text{DNE}$$

$x<0$ | $0<x<4$ | $4<x<6$ | $x>6$

0 4 6

$$f'(x) = \frac{4-x}{x^{1/3}(6-x)^{2/3}}$$

x	$4-x$	$x^{1/3}$	$(6-x)^{2/3}$	$f'(x)$	f
$x<0$	+ve	−ve	+ve	−ve	decr on $(-\infty,0)$
$0<x<4$	+ve	+ve	+ve	+ve	Incr on $(0,4)$
$4<x<6$	−ve	+ve	+ve	−ve	decr on $(4,6)$
$x>6$	−ve	+ve	+ve	−ve	decr on $(6,\infty)$

decr = decreasing

Incr = Increasing

$f(x) = x^{2/3}(6-x)^{1/3}$

Using the first derivative test

(a) f' changes from −ve to +ve near 0, $f(0)=0$ is a local min

(b) f' changes from +ve to −ve near 4, $f(4) = 2^{5/3}$ is a local max

(c) f' does not change sign at 6 so there is no min/max at 6

$$f''(x) = \frac{-8}{x^{4/3}(6-x)^{5/3}}$$

we observe that that $x^{4/3} \geq 0$

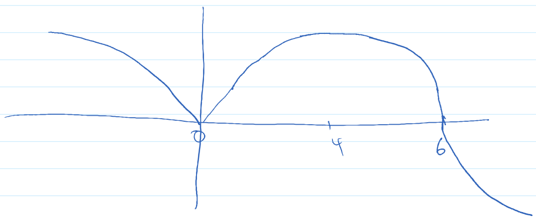
Using concavity test (Curve lies below tangent)

If consider $x<0$, $f''(x) < 0$ f concave down on

If consider $x < 0$, $f''(x) < 0$, f concave downward on $(-\infty, 0)$

If consider $0 < x < 6$, $f''(x) < 0$, f concave downward on $(0, 6)$

If consider $x > 6$, $f''(x) > 0$, f concave upward on $(6, \infty)$ (curve lies above tangent)



$$f(x) = x^{2/3}(6-x)^{1/3}$$

4.4 Indeterminate forms an L'Hôpital's rule (L'Hospital)

L'Hôpital Rule

Suppose f, g are differentiable, $g'(x) \neq 0$ on an open interval I that contains a

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

$\left(\frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \frac{\pm\infty}{0}, \frac{0}{\pm\infty}\right)$

Explanation

(a) $\lim_{x\to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x\to 1} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$

(b) $\lim_{x\to 0} \frac{\sin(x)}{x}$ (since we get an indeterminate here $\frac{0}{0}$)

we use l'Hôpital rule

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

L'Hôpital Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} =$ indeterminate

(where an indeterminate could be $\frac{0}{0}, \frac{\pm\infty}{0}, \frac{0}{\pm\infty}, \frac{\pm\infty}{\pm\infty}$)

then $\lim$ $f(x)$ $\lim$ $f'(x)$

$\lim_{x \to a} g(x)$ ... (indeterminate forms be $\frac{0}{0}, \frac{\pm\infty}{0}, \frac{0}{\pm\infty}, \frac{\pm\infty}{\pm\infty}$)

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Question

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ = indeterminate

and $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ = indeterminate

as long as $g'(x) \neq 0$, we use l'Hôpital rule a second time

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$

Example

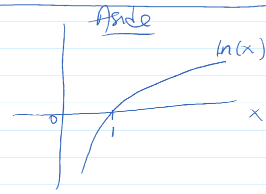
(1) find $\lim_{x \to 1} \frac{\ln(x)}{x-1}$, $\lim_{x \to 1} \ln(1) = 0$

$\lim_{x \to 1} x-1 = 1-1 = 0$

$\frac{0}{0}$

$$\lim_{x \to 1} \frac{\ln(x)}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{\frac{1}{x}}{x} = 1$$

Aside



$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$

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(2) find the $\lim_{x \to \infty} \frac{e^x}{x^2}$ $\left(\frac{\lim_{x\to\infty} e^x}{\lim_{x\to\infty} x^2} = \frac{e^\infty}{\infty} = \frac{\infty}{\infty} \text{ indeterminate} \right)$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \quad \text{(indeterminate)}$$

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d^2}{dx^2}(e^x)}{\frac{d^2}{dx^2}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2} = \frac{\infty}{2} = \infty$$

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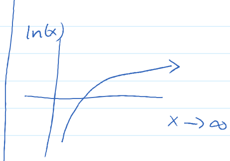
(3) find the $\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \frac{\lim_{x\to\infty} \ln(x)}{\lim_{x\to\infty} \sqrt{x}} = \frac{\infty}{\infty}$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(\sqrt{x})} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}$$

$$= \lim_{x \to \infty} \frac{1}{x} \div \frac{1}{2\sqrt{x}}$$

Aside



if $x \to \infty$

$\ln(x) \to \infty$

$\sqrt{x}$

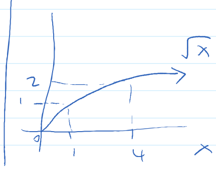
$$= \lim_{x \to \infty} \frac{1}{x} \div \frac{1}{2\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{1}{x} \cdot \frac{2\sqrt{x}}{1}$$

$$= \lim_{x \to \infty} \frac{2\sqrt{x}}{x} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{2\cancel{x}}{\cancel{x}\sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{2}{\sqrt{x}} = \frac{2}{\infty} = 0$$



if $x \to \infty$, $\sqrt{x} \to \infty$