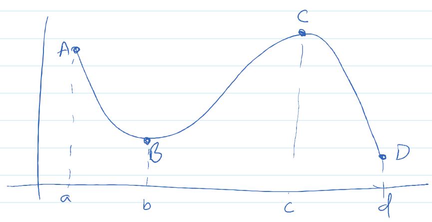
4.3 What does $\boxed{f'}$ tell us about $\boxed{f}$

derivative of a function

help us to Sketch f



f

Q can f' find portions of the domain of f where f is increasing or decreasing

(slope)

(i) $f'(x) > 0$ in B to C, f in increasing on (b, c)

(ii) $f'(x) < 0$ in A to B, f is decreasing on (a, b)

(iii) $f'(x) < 0$ in C to D, f decreasing on $(c, d$

Increasing / Decreasing Test

(i) If $f'(x) > 0$ on an interval, f is increasing on that interval

(ii) If $f'(x) < 0$ on an interval, f is decreasing on that interval

Example

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

find the portion of the domain of f where f is increasing or decreasing

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$f'(x) = 12x(x-2)(x+1)$$

To partition the domain, we need to find the critical points

Set $f'(x) = 0$, solve for x

$12x(x-2)(x+1) = 0$, solve for x

$12x = 0$ or $x - 2 = 0$ or $x + 1 = 0$

$x = 0$ or $x = 2$ or $x = -1$

So the critical points are

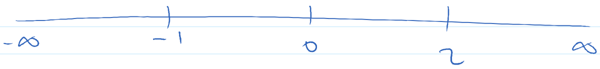
$-1, 0, 2$

aside

Zeroth product property

If $a.b = 0$ then either $a = 0$, $b = 0$, or both are zero

So the critical points are
$-1, 0, 2$



$a = 0$, $b = 0$, or both are zero
a, b are real numbers

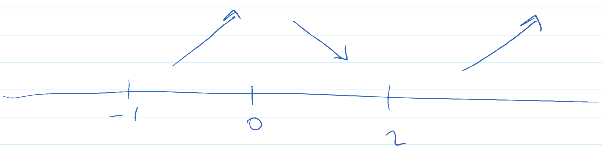
If a, b were not real numbers
say a, b are from $\mathbb{Z}_6 = \{0,1,2,3,4,5\}$
$2 \cdot 3 \equiv 0 \pmod 6$

Next, we use the increasing/decreasing test
$f'(x) = 12x\,(x-2)(x+1)$

x	$12x$	$x-2$	$x+1$	$f'(x)$	f
$x < -1$	$-ve$	$-ve$	$-ve$	$-ve$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-ve$	$-ve$	$+ve$	$+ve$	increasing on $(-1, 0)$
$0 < x < 2$	$+ve$	$-ve$	$+ve$	$-ve$	decreasing on $(0, 2)$
$x > 2$	$+ve$	$+ve$	$+ve$	$+ve$	increasing on $(2, \infty)$

negative $= -ve$
positive $= +ve$

Algebra
$-ve \cdot -ve = +ve$
$+ve \cdot +ve = +ve$
$-ve \cdot +ve = -ve$
$+ve \cdot -ve = -ve$



f has a local max at 0

f has a local min at -1

f has a local min at 2

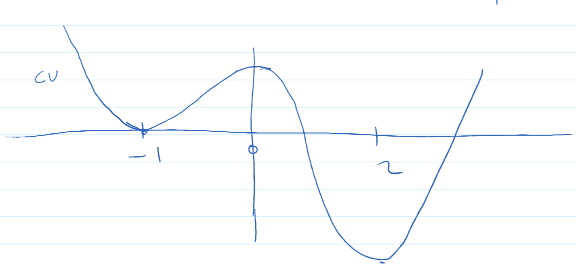
First derivative Test

Suppose c is a critical point

(a) If f' changes from positive to negative at c
f has a local maximum at c

(b) If f' changes from negative to positive at c
f has a local minimum at c

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

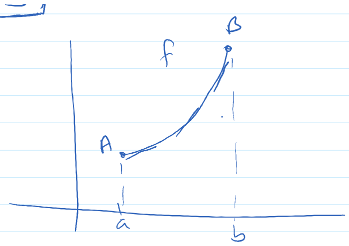


local min at $x = -1$
$f(-1) = 0$

local max at $x = 0$
$f(0) = 5$

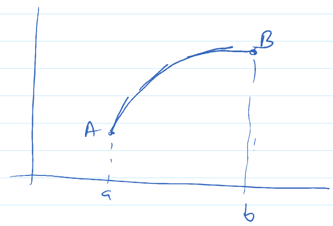
local min at $x = 2$
$f(2) = -27$

$\boxed{\text{Concavity}}$



If a curve lies above tangent

f is called concave upward on (a, b)



If a curve lies below tangent

f is called concave downward on (a, b)

Concavity Test

(a) If $f''(x) > 0$ on an interval, then f concave upward on that interval

(b) If $f''(x) < 0$ on an interval, then f concave downward on that interval

Definition

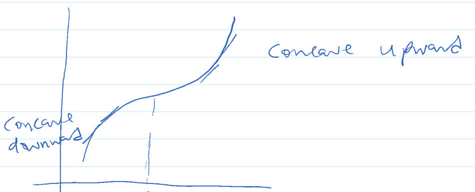
A point P on a curve f is called an inflection point if f is continuous at P and the curve

(i) changes from concave upward to concave downward at P

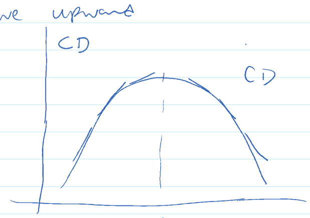
(ii) changes from concave downward to concave upward at P

CD: concave downward

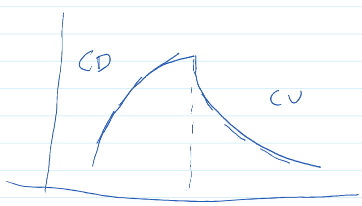
CU: concave upward



P (an inflection point)



a (not an inflection point



P (an inflection point)

Example

Sketch a possible graph of a function f that satisfies the following:

solution

Increasing/decreasing test

(a) $f'(x) > 0$ on $(-\infty, 1)$ $\Rightarrow$ f is increasing on $(-\infty, 1)$

$f'(x) < 0$ on $(1, \infty)$ $\Rightarrow$ f is decreasing on $(1, \infty)$

Concavity test

(b) $f''(x) > 0$ on $(-\infty, -2)$, $(2, \infty)$ — f is concave upward on $(-\infty, -2)$, $(2, \infty)$

$f''(x) < 0$ on $(-2, 2)$ — f is concave downward on $(-2, 2)$

~~Kayode Olumoyin at 11/3/2020 8:14 AM~~

(c) $\lim_{x \to -\infty} f(x) = -2$ and $\lim_{x \to \infty} f(x) = 0$

f has Horizontal asymptote line: $y = -2$

f has Horizontal asymptote line: $y = 0$

Second derivative Test

(converse of the Fermat's theorem + an additional condition)

(a) If $f'(c) = 0$, $f''(c) > 0$ then f has a local min at c

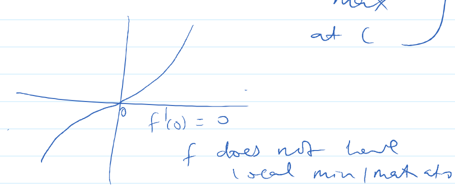
(b) If $f'(c) = 0$, $f''(c) < 0$ then f has a local max at c

Recall Fermat's Theorem

If f has a local min/max at c then c is a critical point ($f'(c) = 0$)

~~Converse of Fermat's theorem~~ ~~[illegible]~~

(If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min/max at c)



The second derivative test is inconclusive when $f''(c) = 0$

Example

Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, point of inflection, local max/min

Solution

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Next find the critical points

Set $f'(x) = 0$, solve for x

$f'(x) = 4x^2(x-3) = 0$

$4x^2(x-3) = 0$

$4x^2 = 0$ or $x - 3 = 0$

$\underline{x=0}$ or $\underline{x=3}$

aside

zeroth product property

If $a \cdot b = 0$ then either $a=0$ or $b=0$ or both are zero

Using 2nd derivative test,

1. $f'(0) = 0$, $f''(0) = 0$ (no information about the critical point 0)

2. $f'(3) = 0$, $f''(3) = 36 > 0$ (there is a local min at 3 $f(3) = -27$)

Using the first derivative test at 0

$f'(x) = 4x^2(x-3)$

x	$4x^2$	$x-3$	$f'(x)$	f
$x<0$	+ve	−ve	−ve	
$0<x<3$	+ve	−ve	−ve	

(we do not get a local min/max at 0)

$f''(x) = 12x(x-2) = 0$ $x = 0$ or 2

Interval	$12x$	$x-2$	$f''(x)$
$(-\infty, 0)$	−ve	−ve	+ve
$(0, 2)$	+ve	−ve	−ve
$(2, \infty)$	+ve	+ve	+ve

Concavity test

$f''(x) > 0$ on $(-\infty, 0)$, concave upward on $(-\infty, 0)$

$f''(x) < 0$ on $(0, 2)$, concave downward on $(0, 2)$

$f''(x) > 0$ on $(2, \infty)$, concave upward on $(2, \infty)$