4.3 What does the derivative tell us about the shape of a graph

Crash Course on College Algebra

[One of the theme of College Algebra, you can sketch $\boxed{\text{polynomial graphs}}$]

(A) Degree of the polynomial (highest exponent on the independent variable)

(i) If a polynomial has degree $n=1$ (line) $f(x) = 2x+3$

(ii) If a polynomial has degree $n=2$ (quadratic) $f(x) = x^2 - x + 3$

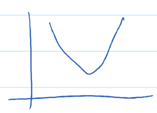
(B) Sign of the leading term ($f(x) = \boxed{ax^2} + bx + c$)

leading term

for example in quadratics,

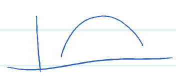
$$f(x) = ax^2 + bx + c$$

If $a > 0$



upward

$a < 0$



downward

(C) Theorem from College Algebra

A polynomial of degree n has

(i) at most n x-intercepts

(ii) at most $n-1$ turning points

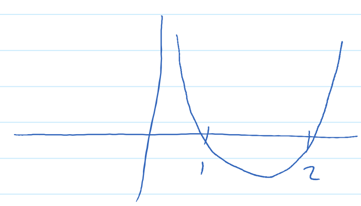
Let use the 3 points above to sketch the graph of $f(x) = x^2 - 3x + 2 = (x-2)(x-1)$

$$f(x) = (x-2)(x-1)$$

(C) f has 2 x-Intercepts, one turning point

(B) graph opens upward (sign of leading term is positive)

(A) shape is quadratic (degree is 2)

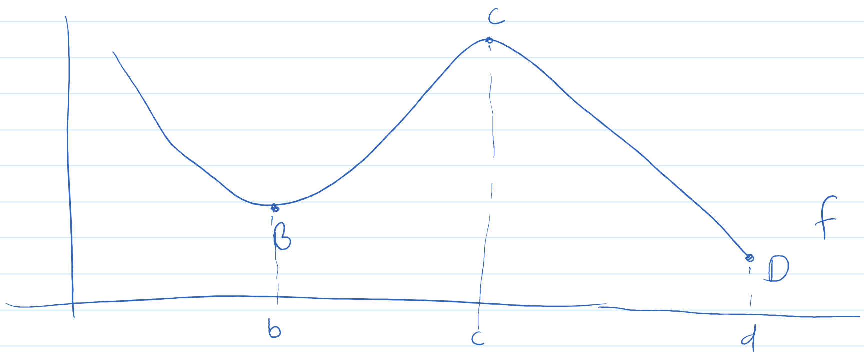


In this section, we want to generalize to any function

And so the question is

Q what does f' tell us about shape of f ?

f' can tell us portions of the domain of f where f



$f'(x)$ (slope) of f in (a,b) is negative, f is decreasing

$f'(x)$ in (b,c) is positive, f is increasing

$f'(x)$ in (c,d) is negative, f decreasing

Increasing/Decreasing Test

(i) If $f'(x) > 0$ on an interval, then f is increasing on that interval

(ii) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Example

find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing / decreasing

Solution

(i) find $f'(x)$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x\left(x^2 - x - 2\right)$$

$$f'(x) = 12x\,(x-2)(x+1)$$

The critical values are found by setting $f'(x) = 0$

$$f'(x) = 12x\,(x-2)(x+1) = 0$$

$$12x = 0 \quad \text{or} \quad x-2=0 \quad \text{or} \quad x+1=0$$

$$x=0 \quad \text{or} \quad x=2 \quad \text{or} \quad x=-1$$

The critical values are $x = -1, 0, 2$

$$\left(f'(0) = 0,\ f'(-1) = 0,\ f'(2) = 0\right)$$

Aside

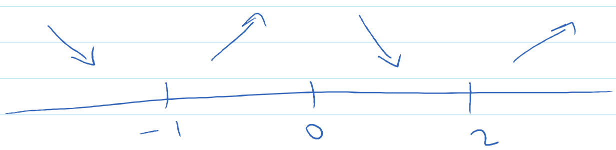
Zeroth Property

if $a \cdot b = 0$

then either $a = 0$ or $b = 0$

or both are zero

The critical values partitions the domain of f



we need to check whether f is increasing / decreasing in the partitions $(-\infty,-1)$, $(-1,0)$, $(0,2)$, $(2,\infty)$

$$f'(x) = 12x\,(x-2)(x+1)$$

inequalities	$12x$	$x-2$	$x+1$	$f'(x)$	f
$-\infty < x < -1$	$-$ve	$-$ve	$-$ve	$-$ve	decreasing on $(-\infty,-1)$
$-1 < x < 0$	$-$ve	$-$ve	$+$ve	$+$ve	increasing on $(-1,0)$
$0 < x < 2$	$+$ve	$-$ve	$+$ve	$-$ve	decreasing on $(0,2)$
$2 < x < \infty$	$+$ve	$+$ve	$+$ve	$+$ve	increasing on $(2,\infty)$

$-$ve = negative

$+$ve = positive

aside

Algebra of signs

$+$ve $\cdot$ $+$ve = $+$ve

$-$ve $\cdot$ $-$ve = $+$ve

$-$ve $\cdot$ $+$ve = $-$ve

$+$ve $\cdot$ $-$ve = $-$ve

The First derivative Test

Recall Fermat's Theorem

[If f has a local maximum or local minimum at c
then c must be a critical number of f]

[but not every critical number of f gives us a
maximum or minimum]

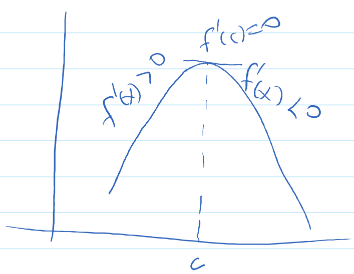
[We need a test that will tell us whether f has
local maximum / minimum at c

<u>First derivative test</u> $\quad (f'(c)=0$ or $f'(c)$ DNE$)$

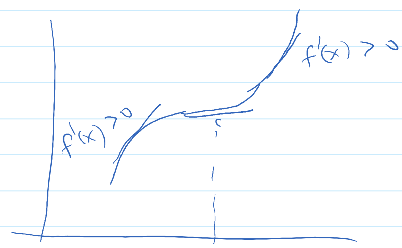
Suppose c is a critical number of a continuous function f

(a) If f' changes from positive to negative at c
then f has a local maximum at c

$f(x) = x^3$

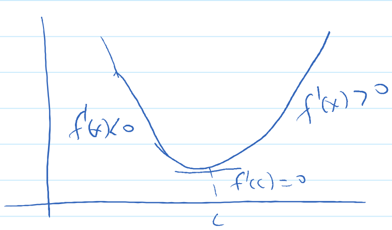


So local maximum at c

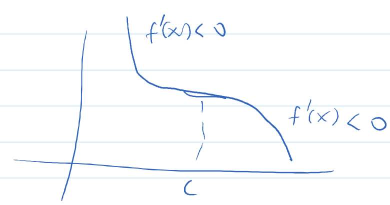


no local max / min at c

(b) If f' changes from negative to positive at c
then f has a local minimum at c



So local minimum at c



no local max / min at c

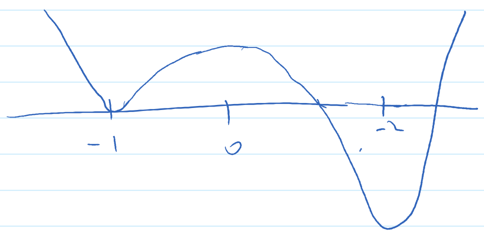
(c) If f' is positive to the left and right of c
or f' is negative to the left and right of c

then f has no local min/max at c

Example

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x(x-2)(x+1)$$



f' changes from -ve to +ve at -1

$f(-1) = 0$ (local minimum at -1)

f' changes from +ve to -ve at 0

$f(0) = 5$ (local maximum at 0)

f' changes from -ve to +ve at 2

$f(2) = -27$ (local minimum at 2)