Exam #3 Practice

#1. $f(t) = e^{mt}\cos(nt)$, find $f'(t)$

$f'(t) = e^{mt}\cdot\frac{d}{dt}(\cos(nt)) + \cos(nt)\cdot\frac{d}{dt}(e^{mt})$

$= e^{mt}\cdot(-n\sin(nt)) + \cos(nt)\cdot m e^{mt}$

$= e^{mt}(-n\sin(nt) + m\cos(nt))$

aside

$\frac{d}{dt}(\cos(nt))$

$u = nt$

$\frac{d}{dt}(\cos(u))$

$\frac{d}{du}(\cos(u))\cdot\frac{du}{dt}$

$-\sin(u)\cdot n$

$= -n\sin(nt)$

$\frac{d}{dt}(e^{mt})$

$v = mt$

$\frac{d}{dt}(e^v)$

$\frac{d}{dv}(e^v)\cdot\frac{dv}{dt}$

$= e^v\cdot m$

$= me^{mt}$

#2. $F(x) = (4x+4)^3(x^2-5x+4)^4$, find $F'(x)$

$F'(x) = (4x+4)^3\cdot\frac{d}{dx}\left[(x^2-5x+4)^4\right] + (x^2-5x+4)^4\cdot\frac{d}{dx}\left[(4x+4)^3\right]$

$= (4x+4)^3\cdot 4(x^2-5x+4)^3(2x-5) +$

$(x^2-5x+4)^4\cdot 3(4x+4)^2\cdot 4$

$= 4(4x+4)^2(x^2-5x+4)^3\left[(4x+4)(2x-5) + (x^2-5x+4)3\right]$

$=$

Aside

$\frac{d}{dx}\left[(4x+4)^3\right]$

set $v = 4x+4$

$\frac{d}{dx}(v^3)$

$\frac{d}{dv}(v^3)\cdot\frac{dv}{dx}$

$3v^2\cdot 4$

$3(4x+4)^2\cdot 4$

Aside

$\frac{d}{dx}\left[(x^2-5x+4)^4\right]$

set $u = x^2-5x+4$

$\frac{d}{dx}(u^4)$

$\frac{d}{du}(u^4)\cdot\frac{du}{dx}$

$4u^3\cdot(2x-5)$

$4(x^2-5x+4)^3(2x-5)$

#3. $y = b^x$ $(b>0,\ b\neq 1)$ tangent line through (x_0, y_0)

pass through $(0,0)$

Show $y_0 = e$

Hint: $x^r = e^{r\ln(x)}$ $\left(e^{\ln x^r} \atop x^r \right)$

$y = b^x$

$\frac{dy}{dx} = b^x \ln b$ slope of the tangent line

$b^{x_0} \ln b$

eqn of the tangent line

$$y - y_0 = b^{x_0} \ln b \, (x - x_0)$$

tangent line pass through $(0,0)$ set $x=0,\ y=0$

$$0 - y_0 = b^{x_0} \ln b \, (0 - x_0)$$

$$y_0 = x_0 b^{x_0} \ln(b)$$

we know $y_0 = b^{x_0}$

$$b^{x_0} = x_0 b^{x_0} \ln(b)$$

recall hint

$x^r = e^{r\ln(x)}$

$\Rightarrow \quad 1 = x_0 \ln(b)$

$\Rightarrow \quad x_0 = \frac{1}{\ln(b)}$

$$y_0 = b^{x_0} = e^{x_0 \ln(b)} = e^{\frac{1}{\ln(b)} \cdot \ln(b)} = e$$

$$\boxed{y_0 = e}$$

#12, 14, 18, 21, 22

#12. find y', y'' by implicit differentiation

$$x^2 + 4y^2 = 4$$

$$\frac{d}{dx}\left(x^2 + 4y^2\right) = \frac{d}{dx}(4)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = 0$$

$$2x + \frac{d}{dx}(4y^2) = 0$$

$$2x + 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = -2x$$

$$y' = \frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$$

Aside

$$\frac{d}{dx}(4y^2)$$

$$\frac{d}{dy}(4y^2)\cdot\frac{dy}{dx}$$

$$8y\cdot\frac{dy}{dx}$$

$$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{-x}{4y}\right)$$

$$= -\frac{1}{4}\cdot\frac{d}{dx}\left(\frac{x}{y}\right)$$

$$= -\frac{1}{4}\cdot\frac{y\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(y)}{y^2}$$

$$= -\frac{1}{4}\cdot\frac{y\cdot 1 - x\cdot\left(\frac{-x}{4y}\right)}{y^2}$$

$$= -\frac{1}{4}\cdot\frac{\left(y + \frac{x^2}{4y}\right)}{y^2}$$

Aside

$$\frac{4y}{4y}\cdot\frac{y}{1} + \frac{x^2}{4y}$$

$$\frac{4y^2 + x^2}{4y}\cdot\frac{1}{y^2}$$

$$x^2 + 4y^2 = 4$$

$$= -\frac{1}{4}\left(\frac{4y^2 + x^2}{4y^3}\right)$$

since $4y^2 + x^2 = 4$

$$= -\frac{1}{\cancel{4}}\left(\frac{\cancel{4}}{4y^3}\right)$$

$$\boxed{y'' = -\frac{1}{4y^3}}$$

#14. Van der Waals Equation

(Ideal gas eqns)

$$\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT$$

P = pressure
V = volume
T = Temperature
R = gas constant
n = # of moles of a gas

Find $\frac{dV}{dP}$

$$PV - Pnb + \frac{n^2a}{V} - \frac{n^3ab}{V^2} = nRT$$

To find $\frac{dV}{dP}$

$$\frac{d}{dP}\left[PV - Pnb + \frac{n^2a}{V} - \frac{n^3ab}{V^2}\right] = \frac{d}{dP}(nRT)$$

$$\frac{d}{dP}(PV) - \frac{d}{dP}(Pnb) + \frac{d}{dP}\left(\frac{n^2a}{V}\right) - \frac{d}{dP}\left(\frac{n^3ab}{V^2}\right) = 0$$

$$P\cdot\frac{dV}{dP} + V\cdot 1 - nb - n^2aV^{-2}\cdot\frac{dV}{dP} + 2n^3abV^{-3}\frac{dV}{dP} = 0$$

$$P\cdot\frac{dV}{dP} - n^2aV^{-2}\frac{dV}{dP} + 2n^3abV^{-3}\frac{dV}{dP} = nb - V$$

$$\frac{dV}{dP}\left(P - n^2aV^{-2} + 2n^3abV^{-3}\right) = nb - V$$

$$\frac{dV}{dP} = \frac{nb - V}{P - n^2aV^{-2} + 2n^3abV^{-3}}\cdot\frac{V^3}{V^3}$$

aside

$\frac{d}{dP}(Pnb)$

$nb\cdot\frac{d}{dP}(P)$

$nb\cdot 1$

$n^2a\cdot\frac{d}{dP}\left(\frac{1}{V}\right)$

$n^2a\cdot\frac{d}{dV}\left(\frac{1}{V}\right)\cdot\frac{dV}{dP}$

$\Downarrow V^{-1}$

$n^2a\cdot(-1\cdot V^{-2})\cdot\frac{dV}{dP}$

$-\frac{n^2a}{V^2}\cdot\frac{dV}{dP}$

$n^3ab\cdot\frac{d}{dP}(V^{-2})$

$n^3ab\cdot\frac{d}{dV}(V^{-2})\cdot\frac{dV}{dP}$

$n^3ab\cdot(-2V^{-3})\cdot\frac{dV}{dP}$

$$\boxed{\frac{dv}{dp} = \frac{v^3(nb - v)}{pv^3 - n^2av + 2n^3ab}}$$

#18, 21, 22

#18.

$f(x) = \log_b(8x^4 - 7)$ for what value of b is $f'(1) = 8$

find $f'(x)$

set $u = 8x^4 - 7$

$$\frac{d}{dx}\left(\log_b(8x^4 - 7)\right)$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_b u)$$

$$\frac{d}{dx}(\log_e(x)) = \frac{1}{x\ln(e)}$$

$$\frac{d}{du}(\log_b u) \cdot \frac{du}{dx}$$

$$\boxed{\frac{d}{du}(\log_b(u)) = \frac{1}{u\ln(b)}}$$

$$f'(x) = \frac{1}{u\ln(b)} \cdot 32x^3$$

$$f'(x) = \frac{32x^3}{(8x^4-7)\ln(b)}$$

given $f'(1) = 8$

$$f'(1) = \frac{32 \cdot 1^3}{(8 \cdot 1^4 - 7)\ln(b)} = 8$$

$$\frac{32}{\ln(b)} = 8$$

$$\Rightarrow \quad 32 = 8\ln(b)$$

$$4 = \ln(b)$$

$$\Rightarrow \quad e^4 = e^{\ln(b)}$$

$$\boxed{e^4 = b}$$

$\boxed{t = 0}$

#21. $s = f(t) = \sin\left(\frac{\pi t}{2}\right)$

need velocity $= v(t) = f'(t)$

need acceleration $a(t) = f''(t)$

(b) find velocity

$f'(t) = \frac{d}{dt}\left(\sin\left(\frac{\pi t}{2}\right)\right)$

Set $u = \frac{\pi t}{2}$

$= \frac{d}{dt}\left(\sin(u)\right)$

$= \frac{d}{du}\left(\sin(u)\right) \cdot \frac{du}{dt}$

$= \cos(u) \cdot \frac{\pi}{2}$

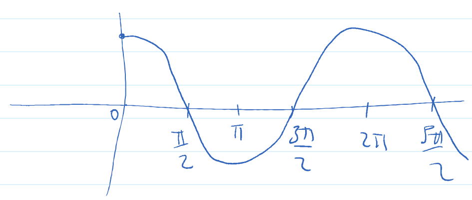
$f'(t) = \frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right)$

(c) find the values of t when the particle is at rest

"find t when $f'(t) = 0$"

find t, $\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right) = 0$

$\Rightarrow \cos\left(\frac{\pi t}{2}\right) = 0$



$\Rightarrow \boxed{t = 2n+1}$ where $n = 0, 1, 2, \ldots$

$n = 0$, $t = 1$

$n = 1$, $t = 3$

$\boxed{t = 2n+1 \quad \text{for non-negative } n}$

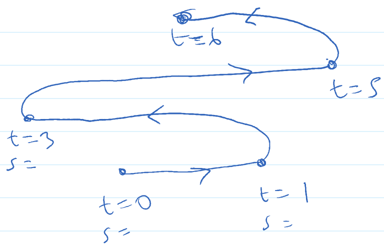
(d) positive direction $v(t) > 0$

$0 < t < 1$, $3 < t < 5$

$(0, 1) \cup (3, 5)$

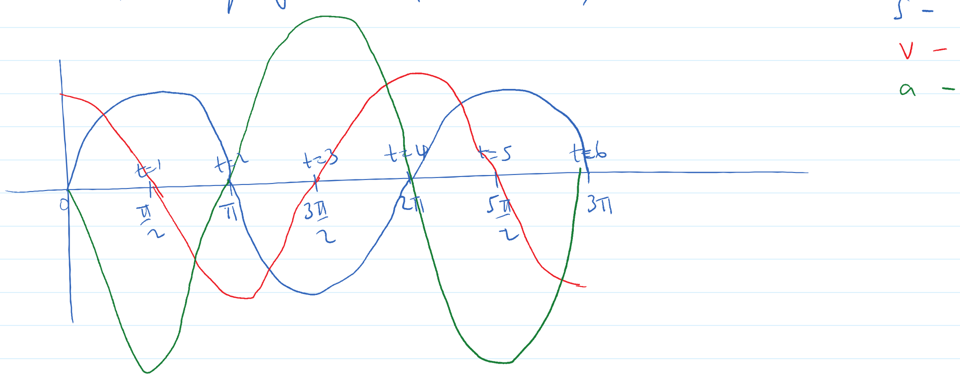
$f(t) = \sin\left(\frac{\pi}{2} t\right)$

(e)



$$|f(1)-f(0)| + |f(3)-f(1)| + |f(5)-f(3)|$$
$$+ |f(6)-f(5)|$$

Sketch the graph of s, v, a



The particle speed up when v, a have same signs

$(1,2) \cup (3,4) \cup (5,6)$

slowing down when v, a have opposite signs

$(0,1) \cup (2,3) \cup (4,5)$

#22. $N_2O_5 \rightarrow 2NO_2 + \frac{1}{2}O_2$

Theorem: If $\frac{d}{dt} = ky$

Then $y(t) = y(0)e^{kt}$

$$-\frac{d}{dt}[N_2O_5] = 0.0006\,[N_2O_5]$$

(a) find $y(t)$, given $y(0) = C$

$k = -0.0006$

$y(t) = y(0)\,e^{-0.0006t}$

$= C\,e^{-0.0006t}$

$$\boxed{y(t) = C\,e^{-0.0006t}}$$

find t,

(b) $y(t) = C\,e^{-0.0006t} = 0.8\,C$

$\Rightarrow \cancel{C}\,e^{-0.0006t} = 0.8\,\cancel{C}$

$$e^{-0.0006t} = 0.8$$

$$\ln\left(e^{-0.0006t}\right) = \ln(0.8)$$

$$-0.0006t = \ln(0.8)$$

$$t = \frac{\ln(0.8)}{-0.0006} \approx \boxed{371.9s}$$