3.8 Exponential Growth and Decay

Differential Equations

[So far, we give you either a function ($y = f(x)$) or (an equation) ($x^2 + y^2 = 1$) and your task was to find the derivative $\frac{dy}{dx}$ (Sometimes you find the second derivative, ...)]

Differential Equations (DiffyQ)

Simple definition of Differential equations

Differential equation is an equation comprising of a function together with some its derivatives

Suppose

$y = f(x)$ you find $\frac{dy}{dx}$ (first derivative), $\frac{d^2y}{dx^2}$ (second derivative), $\frac{d^3y}{dx^3}$

an example of a differential equation can be

$$(*) \quad \boxed{y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} = 0}$$

find y

given $y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

$\frac{d^2y}{dx^2} = -(-)e^{-x} = e^{-x}$

$\frac{d^3y}{dx^3} = -e^{-x}$

$$e^{-x} - e^{-x} + e^{-x} - e^{-x}$$

$$2e^{-x} - 2e^{-x} = 0$$

So a possible solution of $(*)$ is $y = e^{-x}$

$y'' + y = 0$

$y'' = -y$

$\boxed{y = e^x}$

$y' = e^x$

$\boxed{y'' = e^x}$

In a diffy Q – * $\boxed{\text{Uniqueness}}$ (is that solution the only solution)
* $\boxed{\text{existence}}$ (does a solution exist)

In this section (3.8), we want to study an equation comprising a function together with its first derivative

$\boxed{y = f(x)}$

$$\boxed{\frac{dy}{dx} = ky}$$

$\frac{dy}{dx}$: first derivative of the function; y: given function

find y

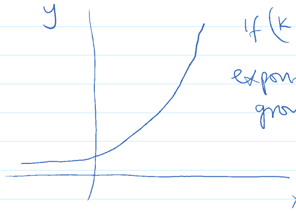
$\boxed{\text{First order differential equation}}$

This is a more befitting

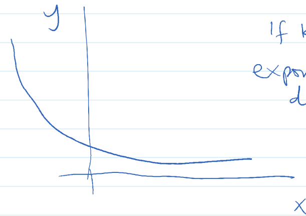
title to this section

Crash Course on exponential functions

$\boxed{y = e^{kx}}$



if $(k>0)$ exponential growth



if $k<0$ exponential decay

I state a theorem that will connect

$$y = e^{kx} \longleftrightarrow \frac{dy}{dx} = ky$$

Theorem

The <u>only</u> Solution of the differential equation $\frac{dy}{dx} = ky$ are the exponential functions

$$y(x) = y(0)\, e^{kx}$$

proof sketch

$\boxed{y = c\, e^{kx}}$ k is a constant   (c is a constant)

$$\frac{dy}{dx} = \frac{d}{dx}\left(c\, e^{kx}\right)$$

$u = x^2$ quadratic
$\frac{du}{dx} = 2x$ linear

$\boxed{u = 2x}$ linear
$\frac{du}{dx} = 2$ $\boxed{\text{constant}}$

Set $u = kx$

$$\boxed{\frac{dy}{dx}} = \frac{d}{dx}\left(c\, e^{kx}\right) = \frac{d}{dx}\left(c\, e^{u}\right) = c \cdot \left(\frac{d}{du} e^{u} \cdot \frac{du}{dx}\right)$$
$$= c \cdot e^{u} \cdot k$$
$$= c \cdot e^{kx} \cdot k$$
$$= k\left(c\, e^{kx}\right)$$
$$= ky$$

$$\boxed{\frac{dy}{dx} = ky \quad \text{when} \quad y = c\, e^{kx}}$$

Population Growth

$(k>0)$ In growth

Chemistry

* Half-life
  Radioactive dating

In decay $(k<0)$

Example 1

$\frac{dP}{dt} = kP$ $P(t)$ = Population at a given time

Example 1

$$\boxed{\frac{dP}{dt} = kP}$$

$P(t)$ = Population at a given time

for population growth $(k > 0)$

given that:

(a) world population in 1950 — 2560 million $= 2.560 \times 10^9$

(b) world population in 1960 — 3040 million $= 3.04 \times 10^9$

assume that growth rate $\frac{dP}{dt}$ $\propto$ Population size P

use our model to estimate world population in 1993 and we will use the model to predict world population in 2025

Solution

Set 1950 (base year) to $t = 0$

So 1960 rep $t = 10$

$P(0) = 2.560 \times 10^9$ ✓

$P(10) = 3.04 \times 10^9$ ✓

according to the theorem, $\boxed{\frac{dP}{dt} = kP}$

then $\boxed{P(t) = P(0)\, e^{kt}}$

$P(t) = 2.560 \times 10^9\, e^{kt}$ (Since $P(0) = 2.560 \times 10^9$)

$$P(10) = 2.560 \times 10^9\, e^{k \cdot 10} = 3.04 \times 10^9$$

Solve for k,

$$e^{k \cdot 10} = \frac{3.04 \times \cancel{10^9}}{2.560 \times \cancel{10^9}}$$

take natural log of both sides

$$\ln e^{k \cdot 10} = \ln\left(\frac{3.04}{2.56}\right)$$

$$k \cdot 10 = \ln\left(\frac{3.04}{2.56}\right)$$

$$k = \frac{1}{10} \ln\left(\frac{3.04}{2.56}\right) \approx 0.017185$$

So

$$P(t) = 2.560 \times 10^9\, e^{0.017185\, t}$$

Since 43 years elapsed from 1950 to 1993

$$1950 \Downarrow t=0 \qquad 1993 \Downarrow t=43$$

To find population in 1993

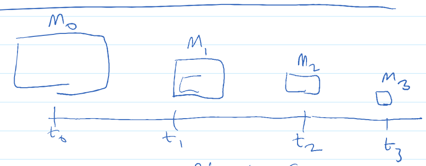
$$P(43) = 2.560 \times 10^9 \, e^{0.017185(43)} \simeq 5.360 \times 10^9$$

Now to predict the population size in 2025

How many years elapsed from 1950 to 2025

$$1950 \Downarrow t=0 \qquad 2025 \Downarrow t=75$$

$$P(75) = 2.560 \times 10^9 \, e^{0.017185(75)} \simeq 9.289 \times 10^9$$



Radioactive decay

$$\frac{dm}{dt} = km$$

m = mass remaining after some initial m_0

$$m(t) = m_0 e^{kt}$$

given that the half-life of radium-226 is 1590 years

(a) A sample of radium-226 has mass 150 mg. Find a formula for the mass of the sample remaining after t years

$$\frac{dm}{dt} = km \quad , \quad m(0) = 150$$

$$m(t) = m(0) e^{kt} = 100 e^{kt}$$

$$m(1590) = \frac{1}{2} m(0) = \frac{1}{2}(100) = 50$$

$$m(t) = 100 e^{kt}$$

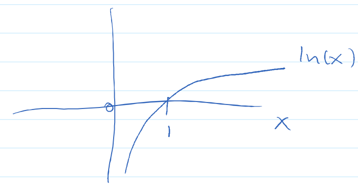
$$m(1590) = 100 e^{k(1590)} = 50$$

$$100 e^{1590k} = 50$$

Aside

$$\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2)$$
$$= 0 - \ln(2)$$

$$e^{1590k} = \frac{1}{2}$$

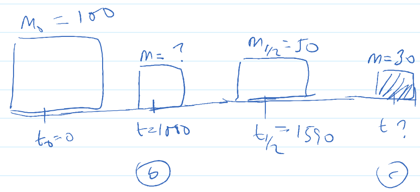


$$\ln e^{1590k} = \ln\left(\tfrac{1}{2}\right)$$

$$1590k = \ln\left(\tfrac{1}{2}\right)$$

$$k = \frac{\ln(1/2)}{1590}$$

$$k = -\frac{\ln(2)}{1590}$$



Carbon dating

$$m(t) = m(0)\, e^{kt}$$

$$\boxed{m(t) = 100\, e^{-\frac{\ln(2)}{1590} t}}$$

(b) Find the mass remaining after 1000 years

$$m(t) = 100\, e^{-\frac{\ln(2)}{1590} t}$$

$$m(1000) = 100\, e^{-\frac{\ln(2)}{1590}\cdot 1000} \approx 65 \text{ mg}$$

(c) find the value of t such that $m = 30$ mg

$$m(t) = 100\, e^{-\frac{\ln(2)}{1590} t} = 30$$

$$100\, e^{-\frac{\ln(2)}{1590} t} = 30$$

$$e^{-\frac{\ln(2)}{1590} t} = \frac{3}{10}$$

$$\ln e^{-\frac{\ln(2)}{1590} t} = \ln\left(\frac{3}{10}\right)$$

$$-\frac{\ln(2)}{1590} t = \ln(3) - \ln(10)$$

$$t = (\ln(3) - \ln(10)) \cdot \left(\frac{1590}{-\ln(2)}\right)$$

$$\approx 2762 \text{ years}$$

3.7 Rates of change in the Natural Sciences (social sciences)

Aside

	Vector	Scalar
f(t)	position	→ distance
f'	velocity	→ speed

Exercise.

The position of a particle

Exercise:

The position of a particle is given by the equation

$s = f(t) = t^3 - 6t^2 + 9t$ (s in meters)
(t in seconds)

	(velocity)	
$f(t)$	Position	→ Distance
$f'(t)$	Velocity	→ Speed
$f''(t)$	—	acceleration

(a) Find the velocity at time t ?

find $f'(t) = 3t^2 - 12t + 9$

$v(t) = 3t^2 - 12t + 9$

(b) Find the velocity after 2 secs

$v(2) = 3(2)^2 - 12(2) + 9 = -3$ m/s

find velocity after 4 secs

$v(4) = 3(4)^2 - 12(4) + 9 = 9$ m/s

(c) when is the particle at rest ?

$f'(t) = v(t) = 0$

$v(t) = 3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

(solve the quadratic eqn for t)

$t^2 - 4t + 3 = 0$

$(t-1)(t-3) = 0$

$t = 1$ or 3

aside

p, q such that

$p + q = -4$

$p \cdot q = 3$

$p = -1, q = -3$

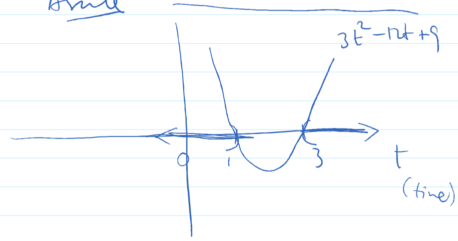
Find when the particle is moving forward

(d) the particle moves in positive direction when $v(t) > 0$

$3t^2 - 12t + 9 > 0$

$3(t-1)(t-3) > 0$

Aside



$t < 1$, $t > 3$

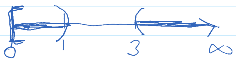
Q when is the graph above the t-axis

Can we write in interval notation

Aside

Can we write in interval notation

Aside

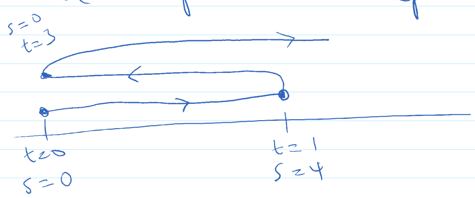


$[0,1), (3,\infty)$

Q above the t-axis

A $t < 1$, $t > 3$

(e) Draw a diagram to rep the motion of this particle



$s = f(t) = t^3 - 6t^2 + 9t$

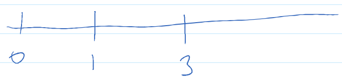
$f(0) = 0$

$f(1) = 4$

$f(3) = 0$

$f(5) = 20$

(f) Find the total distance traveled by the particle during the first 5 secs



$|f(1) - f(0)| = |4 - 0| = 4$

$|f(3) - f(1)| = |0 - 4| = 4$

$|f(5) - f(3)| = |20 - 0| = 20$

total distance $= 28$ m

(g) find acceleration after 4 secs

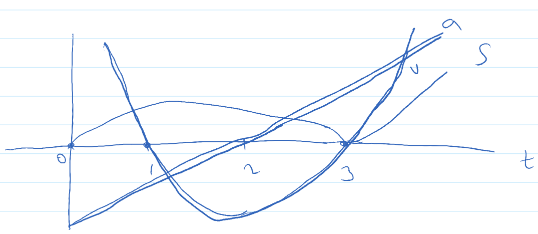
$s = f(t) = t^3 - 6t^2 + 9t$ position

$v(t) = f'(t) = 3t^2 - 12t + 9$ velocity

$a(t) = f''(t) = 6t - 12$ acceleration

$a(4) = 6(4) - 12 = 12 \text{ m/s}^2$

(h) graph s, v, a (use desmos)



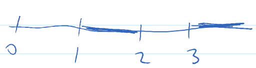
aside

Theorem from college Algebra

A polynomial of degree n has:

(i) atmost n x-intercepts

(ii) atmost $n-1$ turning points



(i) When is the particle speeding up

when v, a are both positive $t > 3 \Rightarrow (3, \infty)$

v, a are both negative $1 < t < 2 \Rightarrow (1, 2)$

When is the particle slowing down

v, a are of opposite signs $(0,1)$, $(2,3)$

3.7 Related Rates

We will go over HW problems

1. (a) If A is the area of a circle with radius r, the circle expands as time passes

find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$

Aside

$A = \pi r^2$

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

(b)

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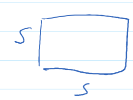
(2) Each sides of a square is increasing at a rate of $\boxed{5\,cm/s}$ at what rate in cm^2/s is the area of the square increasing when the area of the square is $49\ cm^2$

Aside



$A = s^2$

given $A = 49\ cm^2$

$s = 7\ cm$

$\frac{ds}{dt} = 5\ cm/s$

$$\frac{dA}{dt} = \frac{dA}{ds} \cdot \frac{ds}{dt}$$

$$= 2s \frac{ds}{dt}$$

$$\frac{dA}{dt} = 2(7)\,cm \cdot 5\,cm/s \quad = 70\ cm^2/s$$

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(3) The radius of a sphere is increasing at a rate of $3\ mm/s$. How fast is the volume increasing in (mm^3/s) when the diameter is $100\ mm$

Aside

volume of sphere

$V = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = \cancel{3} \cdot \frac{4}{\cancel{3}} \pi r^2$

$= 4\pi r^2$

given

$\frac{dr}{dt} = 3\ mm/s$

$r = \frac{\text{diameter}}{2} = 50\ mm$

$$\frac{dv}{dt} = \frac{dv}{dr} \cdot \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{dr}{dt}$$

$$= 4 \cdot \pi \cdot (50\ mm)^2 \cdot 3\ mm/s$$

$$= 3000\pi\ mm^3/s$$

(4) suppose $4x^2 + 9y^2 = 100$ where x, y are functions of t

(a) If $\frac{dy}{dt} = \frac{1}{3}$, find $\frac{dx}{dt}$ when $x = 4$ and $y = 2$

$$\frac{d}{dt}\left(4x^2 + 9y^2\right) = \frac{d}{dt}(100)$$

$$\frac{d}{dt}(4x^2) + \frac{d}{dt}(9y^2) = 0$$

$$4\frac{d}{dt}(x^2) + 9\frac{d}{dt}(y^2) = 0$$

$$4 \cdot 2x \cdot \frac{dx}{dt} + 9 \cdot 2y \cdot \frac{dy}{dt} = 0$$

$$4 \cdot 2(4)\frac{dx}{dt} + 9 \cdot 2(2)\left(\frac{1}{3}\right) = 0$$

$$32\frac{dx}{dt} + 12 = 0$$

$$\frac{dx}{dt} = \frac{-12}{32} = -\frac{3}{8}$$