

Differential Equations

3.8 Exponential Growth and Decay

So far, we give you either a function $y=f(x)$ or (an equation) and your task was to find the derivative (sometimes you find the second derivative, ...)

Differential Equations (Diffy Q)

Simple definition of Differential equations

Differential equation is an equation comprising of a function together with some its derivatives

Suppose

$y = f(x)$

you find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$
first derivative second derivative

an example of a differential equation can be

(*) $y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} = 0$
 $e^{-x} - e^{-x} + e^{-x} - e^{-x}$
 $2e^{-x} - 2e^{-x} = 0$

find y

guess $y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

$\frac{d^2y}{dx^2} = -(-)e^{-x} = e^{-x}$

$\frac{d^3y}{dx^3} = -e^{-x}$

$y'' + y = 0$
 $y'' = -y$
 $y = e^x$
 $y' = e^x$
 $y'' = e^x$

So a possible solution of (*)

is $y = e^{-x}$

In a diffy Q - uniqueness (is that solution the only solution)
existence (does a solution exist?)

In this section (3.8), we want to study an equation comprising a function together with its first derivative

$y = f(x)$

$\frac{dy}{dx} = ky$

↑ first derivative of the function
 ↑ given function

find y

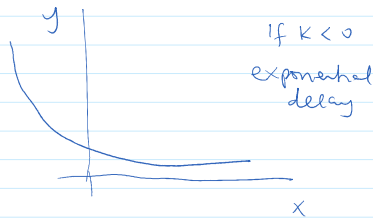
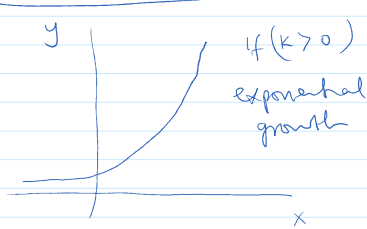
First order differential equation

This is a more befitting

title to this section

Crash Course on exponential functions

$$y = e^{kx}$$



I state a theorem that will connect
 $y = e^{kx} \iff \frac{dy}{dx} = ky$

Theorem

The only solution of the differential equation $\frac{dy}{dx} = ky$ are the exponential functions

$$y(x) = y(0) e^{kx}$$

proof sketch

$$y = c e^{kx}$$

$$\frac{dy}{dx} = \frac{d}{dx} (c e^{kx})$$

k is a constant
 $(c$ is a constant)

$$u = x^2 \quad \text{Then}$$

$$\frac{du}{dx} = 2x \quad \text{Then}$$

$$\Rightarrow u = 2x \quad \text{Then}$$

$$\frac{du}{dx} = 2 \quad \text{Constant}$$

Set $u = kx$

$$\frac{dy}{dx} = \frac{d}{dx} (c e^{kx}) = \frac{d}{dx} (c e^u) = c \cdot \left(\frac{d}{du} e^u \cdot \frac{du}{dx} \right)$$

$$= c \cdot e^u \cdot k$$

$$= c \cdot e^{kx} \cdot k$$

$$= k (c e^{kx})$$

$$= ky$$

$$\frac{dy}{dx} = ky \quad \text{when} \quad y = c e^{kx}$$

Population Growth

$(k > 0)$ In growth

Chemistry

* Half-life

Radioactive dating

In decay $(k < 0)$

Example 1

$$dP = kP$$

$P(t)$ = Population at a given time

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$$\frac{dp}{dt} = kP$$

$P(t)$ = Population at a given time
for population growth ($k > 0$)

given that:

- (a) world population in 1950 - 2560 million = 2.560×10^9
- (b) world population in 1960 - 3040 million = 3.04×10^9

assume that growth rate \propto Population size
 $\frac{dp}{dt} \propto P$

use our model to estimate world population in 1993
and we will use the model to predict world population
in 2025

Solution

Set 1950 (base year) to $t=0$

So 1960 rep $t=10$

$$P(0) = 2.560 \times 10^9 \quad \checkmark$$

$$P(10) = 3.04 \times 10^9 \quad \checkmark$$

according to the theorem,

$$\frac{dp}{dt} = kP$$

then

$$P(t) = P(0) e^{kt}$$

$$P(t) = 2.560 \times 10^9 e^{kt} \quad (\text{since } P(0) = 2.560 \times 10^9)$$

$$P(10) = 2.560 \times 10^9 e^{k \cdot 10} = 3.04 \times 10^9$$

Solve for k ,

$$e^{k \cdot 10} = \frac{3.04 \times 10^9}{2.560 \times 10^9}$$

take natural log of both sides

$$\ln e^{k \cdot 10} = \ln \left(\frac{3.04}{2.56} \right)$$

$$k \cdot 10 = \ln \left(\frac{3.04}{2.56} \right)$$

$$k = \frac{1}{10} \ln \left(\frac{3.04}{2.56} \right) \approx 0.017185$$

So

$$P(t) = 2.560 \times 10^9 e^{0.017185 t}$$

Since 43 years elapsed from 1950 to 1993
 \downarrow \downarrow
 $t=0$ $t=43$

To find population in 1993

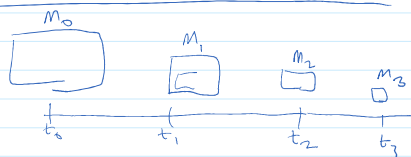
$$P(43) = 2.560 \times 10^9 e^{0.017185(43)} \approx 5.360 \times 10^9$$

Now to predict the population size in 2025

How many years elapsed from 1950 to 2025
 \downarrow \downarrow
 $t=0$ $t=75$

$$P(75) = 2.560 \times 10^9 e^{0.017185(75)} \approx 9.289 \times 10^9$$

Radioactive decay



$$\frac{dm}{dt} = km$$

m = mass remaining after some initial m_0

$$m(t) = m_0 e^{kt}$$

given that the half-life of radium-226 is 1590 years

- (9) A sample of radium-226 has mass 100 mg. Find a formula for the mass of the sample remaining after t years

$$\frac{dm}{dt} = km, \quad m(0) = 100$$

$$m(t) = m(0) e^{kt} = 100 e^{kt}$$

$$m(1590) = \frac{1}{2} m(0) = \frac{1}{2} (100) = 50$$

$$m(t) = 100 e^{kt}$$

$$m(1590) = 100 e^{k(1590)} = 50$$

$$100 e^{1590k} = 50$$

Ans

$$\ln\left(\frac{1}{2}\right) = \ln(1) - \ln(2) \\ = 0 - \ln(2)$$

$$e^{1590k} = \frac{1}{2}$$

$$\ln e^{1590k} = \ln\left(\frac{1}{2}\right)$$

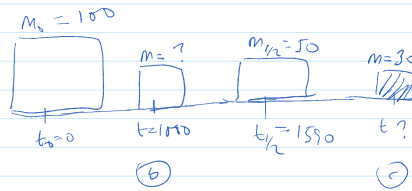
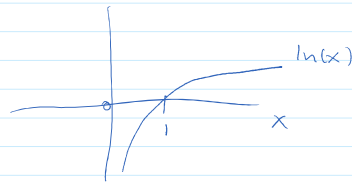
$$1590k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln(1/2)}{1590}$$

$$k = -\frac{\ln(2)}{1590}$$

$$m(t) = m(0)e^{kt}$$

$$m(t) = 100 e^{-\frac{\ln(2)}{1590}t}$$



Carbon dating

(b) Find the mass remaining after 1000 years

$$m(t) = 100 e^{-\frac{\ln(2)}{1590}t}$$

$$m(1000) = 100 e^{-\frac{\ln(2)}{1590} \cdot 1000} \approx 65 \text{ mg}$$

(c) find the value of t such that $m = 30 \text{ mg}$

$$m(t) = 100 e^{-\frac{\ln(2)}{1590}t} = 30$$

$$100 e^{-\frac{\ln(2)}{1590}t} = 30$$

$$e^{-\frac{\ln(2)}{1590}t} = \frac{3}{10}$$

$$\ln e^{-\frac{\ln(2)}{1590}t} = \ln\left(\frac{3}{10}\right)$$

$$-\frac{\ln(2)}{1590}t = \ln(3) - \ln(10)$$

$$t = (\ln(10) - \ln(3)) \cdot \left(\frac{1590}{-\ln(2)}\right)$$

$$\approx 2762 \text{ years}$$

3.7

Rates of Change in the Natural Sciences

(social sciences)

Exercise

The position of a particle

		Aside	
		Vector	Scalar
$f(t)$	Position	→	Distance
v	Velocity	→	Speed

Exercise

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t \quad (s \text{ in meters}) \\ (t \text{ in seconds})$$

	Velocity	Units
$f(t)$	Position	Distance
$f'(t)$	Velocity	Speed
$f''(t)$		acceleration

(a) Find the velocity at time t ?

$$\text{find } f'(t) = 3t^2 - 12t + 9$$

$$v(t) = 3t^2 - 12t + 9$$

(b) Find the velocity after 2 secs

$$v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

find velocity after 4 secs

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

(c) When is the particle at rest?

$$f'(t) = v(t) = 0$$

$$v(t) = 3t^2 - 12t + 9 = 0$$

$$3(t^2 - 4t + 3) = 0$$

$$t^2 - 4t + 3 = 0$$

$$(t-1)(t-3) = 0$$

$$t = 1 \text{ or } 3$$

(Solve the quadratic eqn for t)

aside

p, q such that

$$p+q = -4$$

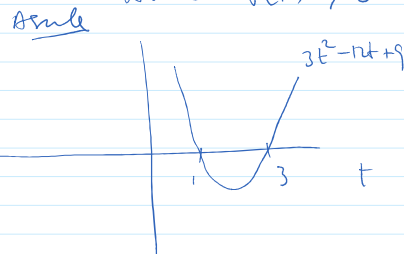
$$p \cdot q = 3$$

$$p = -1, q = -3$$

(d) The particle moves in positive direction when $v(t) > 0$

$$3t^2 - 12t + 9 > 0$$

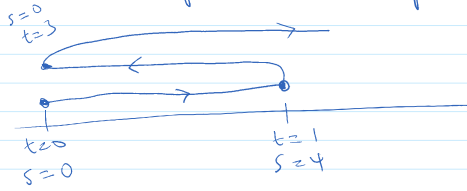
$$3(t-1)(t-3) > 0$$



Q when is the graph above the x-axis

$$A \quad t < 1, \quad t > 3$$

e) Draw a diagram to rep the motion of this particle



$$s = f(t) = t^3 - 6t^2 + 9t$$

f)