3.8 Exponential Growth and Decay

[So far, we give you either a function ($y = f(x)$) or (an equation) ($x^2+y^2=1$) and your task was to find the derivative $\frac{dy}{dx}$ (sometimes you find the second derivative, ...)]

Differential Equations (DiffyQ)

Simple definition of Differential equations

Differential equation is an equation comprising of a function together with some its derivatives

Suppose

$y = f(x)$ you find $\underset{\text{first derivative}}{\frac{dy}{dx}}$, $\underset{\text{second derivative}}{\frac{d^2y}{dx^2}}$, $\frac{d^3y}{dx^3}$

an example of a differential equation can be

$$(*) \quad \boxed{y + \frac{dy}{dx} + \frac{d^2y}{dx^2} + \frac{d^3y}{dx^3} = 0}$$

find y

given $y = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

$\frac{d^2y}{dx^2} = -(-)e^{-x} = e^{-x}$

$\frac{d^3y}{dx^3} = -e^{-x}$

$$e^{-x} - e^{-x} + e^{-x} - e^{-x}$$

$$2e^{-x} - 2e^{-x} = 0$$

So a possible solution of (*) is $y = e^{-x}$

in a diffy Q – uniqueness (is that solution the only solution)
* existence (does a solution exist)

In this section (3.8), we want to study an equation comprising a function together with its first derivative

$\boxed{y = f(x)}$

$$\boxed{\frac{dy}{dx} = ky}$$

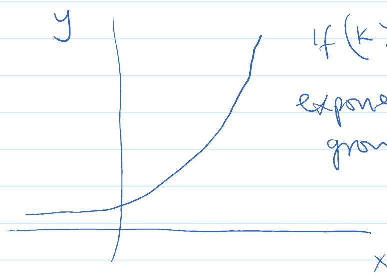
find y

First order differential equation

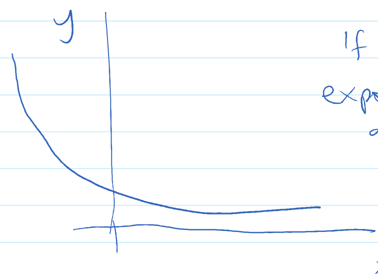
This is a more befitting title to this section

Crash Course on exponential functions

$\boxed{y = e^{kx}}$



if $(k > 0)$ exponential growth



if $k < 0$ exponential decay

I state a theorem that will connect

$$y = e^{kx} \longleftrightarrow \frac{dy}{dx} = ky$$

Theorem

The only solution of the differential equation $\frac{dy}{dx} = ky$ are the exponential functions

$$y(x) = y(0)\, e^{kx}$$

proof sketch

$\boxed{y = c\, e^{kx}}$

k is a constant

(c is a constant)

$u = x^2$ Quadratic

$\frac{du}{dx} = 2x$ linear

$$\boxed{\begin{aligned} u &= 2x \\ \frac{du}{dx} &= 2 \end{aligned}}$$

linear

constant

$$\frac{dy}{dx} = \frac{d}{dx}\left(c\, e^{kx}\right)$$

Set $u = kx$

Set $u = kx$

$$\boxed{\frac{dy}{dx}} = \frac{d}{dx}\left(c\, e^{kx}\right) = \frac{d}{dx}\left(c\, e^{u}\right) = c \cdot \left(\frac{d}{du} e^{u} \cdot \frac{du}{dx}\right)$$

$$= c \cdot e^{u} \cdot k$$

$$= c \cdot e^{kx} \cdot k$$

$$= k\left(c\, e^{kx}\right)$$

$$= k y$$

$$\boxed{\frac{dy}{dx} = k y \quad \text{when} \quad y = c\, e^{kx}}$$

Population Growth

Chemistry

* Half-life

Radioactive dating

Example 1

$\frac{dP}{dt} = kP$ $\quad$ $P(t)$ = Population at a given time for population growth $(k > 0)$

given that:

(a) world population in 1950 — 2560 million $= 2.560 \times 10^{9}$

(b) world population in 1960 — 3040 million $= 3.04 \times 10^{9}$

assume that growth rate $\propto$ Population size

$$\frac{dP}{dt} \propto P$$

use our model to estimate world population in 1993 and we will use the model to predict world population in 2025

Solution

Set 1950 (base year) to $t=0$

So 1960 rep $t=10$

$P(0) = 2.560 \times 10^9$ ✓

$P(10) = 3.04 \times 10^9$ ✓

according to the theorem, $\frac{dP}{dt} = kP$

then $P(t) = P(0)e^{kt}$

$P(t) = 2.560 \times 10^9 e^{kt}$ (Since $P(0) = 2.560 \times 10^9$)

$P(10) = 2.560 \times 10^9 e^{k \cdot 10} = 3.04 \times 10^9$

Solve for k,

$$e^{k \cdot 10} = \frac{3.04 \times \cancel{10^9}}{2.560 \times \cancel{10^9}}$$

take natural log of both sides

$$\ln e^{k \cdot 10} = \ln\left(\frac{3.04}{2.56}\right)$$

$$k \cdot 10 = \ln\left(\frac{3.04}{2.56}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{3.04}{2.56}\right) \approx 0.017185$$

so $P(t) = 2.560 \times 10^9 e^{0.017185\, t}$

Since 43 years elapsed from 1950 to 1993

$$1950 \Downarrow t=0 \qquad 1993 \Downarrow t=43$$

To find population in 1993

To find population in 1993

$P(43) = 2.560 \times 10^9 \, e^{0.017185(43)} \approx 5.360 \times 10^9$

Now to predict the population size in 2025

How many years elapsed from 1950 to 2025

$1950 \Downarrow t = 0$ $\qquad$ $2025 \Downarrow t = 75$

$P(75) = 2.560 \times 10^9 \, e^{0.017185(75)} \approx 9.289 \times 10^9$