3.6 Derivatives of Logarithmic and Inverse Trigonometric functions

1. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$ $\quad(\ln b = \log_e b)$

2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Exercises

① $\frac{d}{dx}(\ln(\sin(x)))$

Set $u = \sin(x)$, $\frac{du}{dx} = \cos(x)$

$$\frac{d}{dx}\left(\ln(\sin(x))\right) = \frac{d}{dx}(\ln(u)) = \frac{d}{du}(\ln(u)) \cdot \frac{du}{dx}$$

$$= \frac{1}{u} \cdot \cos(x)$$

$$= \frac{1}{\sin(x)} \cdot \cos(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$$

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② $\frac{d}{dx}(\sqrt{\ln(x)})$

Set $u = \ln(x)$

$$\frac{d}{dx}\left(\sqrt{\ln(x)}\right) = \frac{d}{dx}(\sqrt{u}) = \frac{d}{du}(\sqrt{u}) \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot \frac{1}{x}$$

$$= \frac{1}{2\sqrt{\ln(x)}} \cdot \frac{1}{x}$$

$$= \frac{1}{2x\sqrt{\ln(x)}}$$

Aside

$$\frac{d}{du}(\sqrt{u}) = \frac{d}{}$$

$$= \frac{1}{2}u^{\frac{1}{2}-1}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{u}}$$

$u = \ln(x)$

$$\frac{du}{dx} = \frac{d}{dx}(\ln(x))$$

$$= \frac{1}{x}$$

Logarithmic Differentiation

(use the properties of logarithms to simplify 'complicated' derivatives)

Crash Course on Logarithms

1. purpose – make complex operations become simpler versions

1. logarithms makes multiplication behave like addition

$$\log(A \cdot B) = \log(A) + \log(B)$$

show

compute $2 \cdot 3$

instead of doing $2 \cdot 3$, I will take log

$$\log(2 \cdot 3) = \log(2) + \log(3)$$
$$= 0.301 + 0.477$$
$$\log(2 \cdot 3) = 0.778$$
$$10^{\log(2 \cdot 3)} = 10^{0.778}$$
$$2 \cdot 3 = 5.998 \approx 6$$

(2) logarithms make Exponents behave like multiplication

$$\log x^r = r \cdot \log x$$

(3) Logarithms make division behave like subtraction

$$\log\left(\frac{A}{B}\right) = \log(A) - \log(B)$$

(4) logarithms make 'Big' numbers behave like 'small' numbers

$$\log 10 = 1$$
$$\log 100 = 2$$
$$\log 1000 = 3$$
$$\log 10000 = 4$$

Exercise

use logarithms properties to find the derivative of

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

step 1. Take natural log of both sides

$$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$$= \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln(3x+2)^5$$

$$= \ln x^{3/4} + \ln\sqrt{x^2+1} - \ln(3x+2)^5$$

$$= \frac{3}{4}\ln(x) + \ln(x^2+1)^{1/2} - 5\ln(3x+2)$$

$$\ln(y) = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

step 2 take derivative of both sides

$$\frac{d}{dx}(\ln(y)) = \frac{d}{dx}\left(\frac{3}{4}\ln(x)\right) + \frac{d}{dx}\left(\frac{1}{2}\ln(x^2+1)\right) - \frac{d}{dx}\left(5\ln(3x+2)\right)$$

$u = x^2+1$ $\quad$ $v = 3x+2$

$$\frac{d}{dy}(\ln(y))\cdot\frac{dy}{dx} = \frac{3}{4}\frac{d}{dx}(\ln(x)) + \frac{1}{2}\frac{d}{dx}\left(\ln(x^2+1)\right) - 5\frac{d}{dx}\left(\ln(3x+2)\right)$$

$\frac{1}{2}\frac{d}{du}(\ln(u))\frac{du}{dx}$

$\frac{1}{2}\cdot\frac{1}{u}\cdot 2x$

$\frac{1}{\cancel{2}}\cdot\frac{1}{(x^2+1)}\cdot\cancel{2}x$

$\frac{x}{x^2+1}$

$5\cdot\frac{d}{dv}(\ln(v))\cdot\frac{dv}{dx}$

$5\cdot\frac{1}{v}\cdot 3$

$5\cdot\frac{1}{(3x+2)}\cdot 3 = \frac{15}{(3x+2)}$

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{x}{x^2+1} + \frac{15}{3x+2}$$

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2+1} + \frac{15}{3x+2}\right)$$

$$\frac{\quad}{dx} = y\left(\frac{\quad}{4x} \quad \frac{\quad}{x^2+1} \quad \frac{\quad}{3x+2}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} + \frac{15}{3x+2}\right)}$$

More Examples (use properties logarithms to help us find derivative)

differentiate $y = x^{\sqrt{x}}$

Step 1 take natural logarithm of both sides

$$\ln(y) = \ln x^{\sqrt{x}}$$

$$\ln(y) = \sqrt{x}\ \ln(x)$$

Step 2 differentiate both sides

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(\sqrt{x}\ \ln(x)\right)$$

$$\frac{d}{dy}\left(\ln(y)\right)\cdot\frac{dy}{dx} = \sqrt{x}\cdot\frac{d}{dx}\left(\ln(x)\right) + \ln(x)\cdot\frac{d}{dx}\left(\sqrt{x}\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sqrt{x}\cdot\frac{1}{x} + \ln(x)\cdot\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = y\left(\frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}}\right)$$

$$\boxed{\frac{dy}{dx} = x^{\sqrt{x}}\left(\frac{\sqrt{x}}{x} + \frac{\ln(x)}{2\sqrt{x}}\right)}$$

Aside

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right)$$

$$= \frac{1}{2}x^{\frac{1}{2}-1}$$

$$= \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}}$$

Read on John Napier and his invention of logarithms

* Nowadays, if we say log 2 (we mean $\log_{10} 2 \approx 0.301$)

find out the original base John Napier used find out the original base John Napier used to construct his table of logarithms

Exercise from Homework 3.6

Differentiate f and find the domain of f

$$f(x) = \sqrt{3 + \ln(x)}$$

(a) find $f'(x)$

$$f(x) = \sqrt{3+\ln(x)} = (3+\ln(x))^{1/2}, \quad \text{Set } u = 3 + \ln(x)$$

$$\frac{d}{dx}\left((3+\ln(x))^{1/2}\right) = \frac{d}{dx}(u^{1/2}) = \frac{d}{du}(u^{1/2}) \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot \frac{1}{x}$$

$$= \frac{1}{2x\sqrt{3+\ln(x)}}$$

(b) Domain of f

$$= \{x \mid 3 + \ln(x) \geq 0\}$$

$$= \{x \mid \ln(x) \geq -3\}$$

$$= \{x \mid e^{\ln(x)} \geq e^{-3}\}$$

$$= \{x \mid x \geq e^{-3}\} = [e^{-3}, \infty)$$