3.4 The Chain Rule

Examples $F(x) = \sqrt{x^2+1}$

$$F(x) = (f \circ g)(x) = f(g(x))$$

Approach I

$$f(x) = \sqrt{x}, \quad g(x) = x^2+1$$

$$F'(x) = f'(g(x))\, g'(x)$$

Approach II

$$y = F(x) = \sqrt{x^2+1}$$

$$y = \sqrt{u} \qquad \text{Set } u = x^2+1$$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 2x$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

More Exercises

Differentiate $y = (2x+1)^5 (x^3-x+1)^4$

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}\left((x^3-x+1)^4\right) + (x^3-x+1)^4 \frac{d}{dx}\left((2x+1)^5\right)$$

$$\boxed{\frac{dy}{dx} = (2x+1)^5 4(x^3-x+1)^3(3x^2-1) + (x^3-x+1)^4 \cdot 5(2x+1)^4 \cdot 2}$$

Aside

$$\frac{d}{dx}\left((x^3-x+1)^4\right) = \frac{d}{dx}(u^4) = 4 \cdot u^3 \cdot \frac{du}{dx} = 4(x^3-x+1)^3(3x^2-1)$$

Set $u = x^3-x+1$

$$\frac{du}{dx} = 3x^2-1$$

$$\frac{d}{dx}\left((2x+1)^5\right) = \frac{d}{dx}(v^5) = 5v^4 \cdot \frac{dv}{dx} = 5v^4 \cdot 2$$

$$= 5(2x+1)^4 \cdot 2$$

Set $v = 2x+1$

$$\frac{dv}{dx} = 2$$

$\frac{dv}{dx} = 2$

Exercise

Differentiate $y = e^{\sin(x)}$ (use chain Rule)

Set $u = \sin(x)$

$y = e^u$, $\frac{dy}{du} = e^u$

$u = \sin x$, $\frac{du}{dx} = \cos(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u \cdot \cos(x)$$
$$= e^{\sin(x)} \cdot \cos(x)$$

Aside

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$f'(0) = \lim_{h\to\infty} \frac{2^h - 1}{h} \approx 0.693$

$f(x) = 2^x$

$f'(x) = 2^x \cdot \lim_{h\to 0} f'(0)$

$f(x) = e^x$

$f'(x) = e^x \cdot 1$

$f'(0) = 1$

$\lim_{h\to 0} f'(0) = \lim_{h\to 0} \frac{e^h - 1}{h} \approx 1$

Rule.
$$\frac{d}{dx}(b^x) = b^x \ln(b)$$

where $\ln(x)$ = natural log $= \log_e(x)$

Pf. we see $b^x = e^{\ln(b) \cdot x}$

$e^{\ln(e)} = e$

$e^{\ln(b)} = b$

$e^{\ln(x)} = x$

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{\ln(b) \cdot x}\right)$$

Set $u = \ln(b) \cdot x$

$$= \frac{d}{dx}(e^u)$$
$$= \frac{d}{du}(e^u) \cdot \frac{du}{dx}$$
$$= e^u \cdot \frac{d}{dx}(\ln(b) \cdot x)$$
$$= e^{\ln(b) \cdot x} \cdot \ln(b)$$

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln(b)}$$

$\ln(e) = 1$

Exercise

① $\frac{d}{dx}(e^x) = e^x \ln(e)$
$= e^x$

② $\frac{d}{dx}(2^x) = 2^x \ln(2)$

Rule

$$\boxed{\frac{d}{dx}(b^x) = b^x \cdot \ln(b)}$$

$$\frac{d}{du}(5^u) = 5^u \cdot \ln(5)$$

③ $\frac{d}{dx}\left(5^{x^2}\right) = \frac{d}{dx}\left(5^u\right) = \frac{d}{du}\left(5^u\right)\cdot\frac{du}{dx}$

set $u = x^2$

$= 5^u \cdot \ln(5)\cdot(2x)$

$= 5^{x^2}\cdot \ln(5)\cdot(2x)$

3.5 Implicit differentiation

So far, we have been working with functions

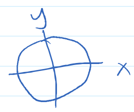
$$\boxed{y = f(x)}$$

a function that depends on x

To differentiate things means to find $\frac{dy}{dx}$

How do you differentiate an equation of the form below

$$\boxed{x^2 + y^2 = 1}$$



(equation of a circle)

Q. How do you find $\frac{dy}{dx}$?

A. we will use the $\boxed{\text{Chain rule}}$ 3.4

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}, \quad \boxed{\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}}$$

Aside

$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$

$= 2y$

$x^2+y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

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② differentiate $x^3 + y^3 = 6xy$

Aside

$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx}$

(a) differentiate $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(6xy)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\cdot\frac{d}{dx}(y) + y\cdot\frac{d}{dx}(6x)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + y\cdot 6$$

$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 6x\right) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{\cancel{3}(2y - x^2)}{\cancel{3}(y^2 - 2x)}$$

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx} = 3y^2\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}(y) = \frac{d}{dy}(y)\cdot\frac{dy}{dx} = 1\cdot\frac{dy}{dx}$$

(b) find the tangent line to $x^3 + y^3 = 6xy$ at the point $(3,3)$

at $(x=3, y=3)$, $\frac{dy}{dx} = \frac{2(3) - 3^2}{3^2 - 2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$

equation of a line $\qquad m = \frac{dy}{dx}$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -1(x - 3)$$

$$y - 3 = -x + 3$$

$$\boxed{y + x = 6}$$

equation of the tangent line to $x^3 + y^3 = 6xy$ at point $(3,3)$

(c) at what point in the first quadrant is the tangent line horizontal?

Ans The tangent line is horizontal when

$$\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x} = 0 \qquad (y^2 - 2x \neq 0)$$

so $2y - x^2 = 0$

$2y = x^2$

$$\boxed{y = \frac{1}{2}x^2}$$

to get values for x, sub into $x^3 + y^3 = 6xy$

$$x^3 + \left(\tfrac{1}{2}x^2\right)^3 = 6x\left(\tfrac{1}{2}x^2\right)$$

$$x^3 + \tfrac{1}{8}x^6 = 3x^3$$

$$\tfrac{1}{8}x^6 = 2x^3$$

$$x^6 = 16x^3$$

$$x^3 = 16$$

$$x = \sqrt[3]{16} = 16^{1/3} = (2^4)^{1/3} = 2^{4/3}$$

$x \approx \boxed{?}$ using your calculator

3.6 Derivative of logarithmic function and inverse Trigonometric functions

Recall

$$\boxed{\frac{d}{dx}(b^x) = b^x \ln b}$$

we will use the above result to help us find the derivative of $\log_b x$

$$\boxed{\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}}$$

Pf.

$$b^y = x$$

now differentiate both sides with respect to x

$$\frac{d}{dx}(b^y) = \frac{d}{dx}(x)$$

$$\frac{d}{dy}(b^y) \cdot \frac{dy}{dx} = 1$$

$$b^y \ln b \, \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$

Aside

$\log_b x = y$

make both sides exponent of b

$$b^{\log_b x} = b^y$$

$$x = b^y$$

$$\frac{d}{dx}(b^y) = \frac{d}{dy}(b^y) \cdot \frac{dy}{dx}$$

Recall

$$\frac{d}{dy}(b^y) = b^y \ln b$$

Examples

$$\frac{d}{dx}(\log_2 x) = \frac{1}{x \ln 2}$$

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}(\log_e x) = \frac{1}{x \ln e} = \frac{1}{x}$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

Exercise

1. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

differentiate

$y = \log_e(x^3+1)$

$\Updownarrow$

$$y = \ln(x^3+1)$$

goal here is to find $\frac{dy}{dx}$

Set $u = x^3 + 1$, rewrite $y = \ln(x^3+1) = \ln(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{du} = \frac{d}{du}(\ln(u)) = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{1}{u} \cdot 3x^2$$

$$= \frac{3x^2}{x^3+1}$$

$\ln(x) = \log_e x$ (natural logarithm)

e = euler number

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

$e = 2.718\ldots$ (irrational number)

$$\boxed{y = 2^x \qquad \frac{dy}{dx} = 2^x \ln 2}$$

$$\boxed{y = e^x \qquad \frac{dy}{dx} = e^x}$$

The most beautiful equation in mathematics

$$\boxed{e^{i\pi} = -1}, \quad i = \sqrt{-1}$$

Question from 3.5 Homework

(1) Find dy by implicit differentiation

Question from 3.5 Homework

① find $\frac{dy}{dx}$ by implicit differentiation

$$\cos(x+y) = \sin(x) + \sin(y)$$

$$\frac{d}{dx}\left(\cos(x+y)\right) = \frac{d}{dx}\left(\sin(x) + \sin(y)\right)$$

$$\frac{d}{dx}\left(\cos(x+y)\right) = \frac{d}{dx}\left(\sin(x)\right) + \frac{d}{dx}\left(\sin(y)\right)$$

$$-\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right) = \cos(x) + \cos(y)\frac{dy}{dx}$$

$$-\sin(x+y) - \sin(x+y)\frac{dy}{dx} = \cos(x) + \cos(y)\frac{dy}{dx}$$

$$-\sin(x+y)\frac{dy}{dx} - \cos(y)\frac{dy}{dx} = \sin(x+y) + \cos(x)$$

$$-\left[\sin(x+y) + \cos(y)\right]\frac{dy}{dx} = \sin(x+y) + \cos(x)$$

$$\frac{dy}{dx} = -\frac{\sin(x+y) + \cos(x)}{\sin(x+y) + \cos(y)}$$

Aside

$$\frac{d}{dx}\left(\cos(x+y)\right)$$

Set $u = x+y$

so

chain rule

$$\frac{d}{dx}\left(\cos(u)\right) = \frac{d}{du}(\cos(u))\cdot\frac{du}{dx}$$

$$= -\sin(u)\cdot\left(1+\frac{dy}{dx}\right)$$

$$= -\sin(x+y)\cdot\left(1+\frac{dy}{dx}\right)$$

$$\frac{d}{dx}\left(\sin(y)\right) = \frac{d}{dy}(\sin(y))\cdot\frac{dy}{dx}$$

$$= \cos(y)\frac{dy}{dx}$$

More question from 3.5 Homework

Find $\frac{dy}{dx}$ by implicit differentiation

$$\cos(xy) = \sin(x+y)$$

$$\frac{d}{dx}\left(\cos(xy)\right) = \frac{d}{dx}\left(\sin(x+y)\right)$$

$$-\sin(xy)\cdot\left(x\frac{dy}{dx}+y\right) = \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$$

$$-x\sin(xy)\frac{dy}{dx} - y\sin(xy) = \cos(x+y) + \cos(x+y)\frac{dy}{dx}$$

$$-x\sin(xy)\frac{dy}{dx} - \cos(x+y)\frac{dy}{dx} = y\sin(xy) + \cos(x+y)$$

$$-\left[x\sin(xy) + \cos(x+y)\right]\frac{dy}{dx} = y\sin(xy) + \cos(x+y)$$

$$\frac{dy}{dx} = -\frac{y\sin(xy) + \cos(x+y)}{x\sin(xy) + \cos(x+y)}$$

Aside,

Set $u = xy$

$$\frac{du}{dx} = x\cdot\frac{dy}{dx} + y\cdot\frac{d}{dx}(x) = x\frac{dy}{dx} + y$$

So, we have

$$\frac{d}{dx}\left(\cos(xy)\right) = \frac{d}{dx}\left(\cos(u)\right)$$

chain rule

$$= \frac{d}{du}(\cos(u))\cdot\frac{du}{dx}$$

$$= -\sin(u)\cdot\left(x\frac{dy}{dx}+y\right)$$

$$= -\sin(xy)\left(x\frac{dy}{dx}+y\right)$$

Aside

Set $v = x+y$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\frac{d}{dx}\left(\sin(x+y)\right) = \frac{d}{dx}\left(\sin(v)\right)$$

chain rule

$$= \frac{d}{dv}(\sin(v))\cdot\frac{dv}{dx}$$

$$= \cos(v)\cdot\left(1+\frac{dy}{dx}\right)$$

$$= \cos(x+y)\cdot\left(1+\frac{dy}{dx}\right)$$