3.4 The Chain Rule

Examples $F(x) = \sqrt{x^2+1}$

$$F(x) = (f \circ g)(x) = f(g(x))$$

Approach I

$$f(x) = \sqrt{x}, \quad g(x) = x^2+1$$

$$F'(x) = f'(g(x))\, g'(x)$$

Approach II

$$y = F(x) = \sqrt{x^2+1}$$

$y = \sqrt{u}$ Set $u = x^2+1$

$$F'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 2x$$

$$= \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

More Exercises

Differentiate $y = (2x+1)^5 (x^3 - x + 1)^4$

$$\frac{dy}{dx} = (2x+1)^5 \frac{d}{dx}\left((x^3-x+1)^4\right) + (x^3-x+1)^4 \frac{d}{dx}\left((2x+1)^5\right)$$

$$\boxed{\frac{dy}{dx} = (2x+1)^5\, 4(x^3-x+1)^3(3x^2-1) + (x^3-x+1)^4 \cdot 5(2x+1)^4 \cdot 2}$$

$$\ldots^4) = \frac{d}{dx}(u^4) = 4 \cdot u^3 \cdot \frac{du}{dx} = 4(x^3-x+1)^3(3x^2-1)$$

Aside

$$\frac{d}{dx}\left((x^3 - x + 1)^4\right) = \frac{d}{dx}(u^4) = 4 \cdot u^3 \cdot \frac{du}{dx} = 4(x^3 - x + 1)^3(3x^2 - 1)$$

Set $u = x^3 - x + 1$

$$\frac{du}{dx} = 3x^2 - 1$$

$$\frac{d}{dx}\left((2x+1)^5\right) = \frac{d}{dx}(v^5) = 5v^4 \cdot \frac{dv}{dx} = 5v^4 \cdot 2$$

$$= 5(2x+1)^4 \cdot 2$$

Set $v = 2x + 1$

$$\frac{dv}{dx} = 2$$

Exercise

Differentiate $y = e^{\sin(x)}$ (use chain Rule)

Set $u = \sin(x)$

$y = e^u$, $\frac{dy}{du} = e^u$

$u = \sin x$, $\frac{du}{dx} = \cos(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= e^u \cdot \cos(x)$$

$$= e^{\sin(x)} \cdot \cos(x)$$

Aside

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

$f'(0) = \frac{2^h - 1}{h}$, $h \to \infty$ ≈ 0.693

$f(x) = 2^x$

$f'(x) = 2^x \cdot \lim_{h \to \infty} f'(0)$

$f(x) = e^x$

$f'(x) = e^x \cdot 1$

$f'(0) = 1$

$\lim_{h \to 0} f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} \approx 1$

Rule.

$$\frac{d}{dx}(b^x) = b^x \ln(b)$$

where

$\ln(x)$ = natural log

$= \log_e(x)$

Pf. we see $b^x = e^{\ln(b) \cdot x}$

$$e^{\ln(e)} = e$$

$$e^{\ln(b)} = b$$

$$e^{\ln(x)} = x$$

$$\frac{d}{dx}(b^x) = \frac{d}{dx}\left(e^{\ln(b) \cdot x}\right)$$

Set $u = \ln(b) \cdot x$

$= d\ (e^u)$

set $u = \ln(b) \cdot x$

$$= \frac{d}{dx}\left(e^u\right)$$

$$= \frac{d}{du}\left(e^u\right) \cdot \frac{du}{dx}$$

$$= e^u \cdot \frac{d}{dx}\left(\ln(b) \cdot x\right)$$

$$= e^{\ln(b) \cdot x} \cdot \ln(b)$$

$$\boxed{\frac{d}{dx}\left(b^x\right) = b^x \ln(b)}$$

$\ln(e) = 1$

Exercise

① $\frac{d}{dx}\left(e^x\right) = e^x \ln(e)$
$= e^x$

Rule

$$\boxed{\frac{d}{dx}\left(b^x\right) = b^x \cdot \ln(b)}$$

$$\frac{d}{du}\left(5^u\right) = 5^u \cdot \ln(5)$$

② $\frac{d}{dx}\left(2^x\right) = 2^x \ln(2)$

③ $\frac{d}{dx}\left(5^{x^2}\right) = \frac{d}{dx}\left(5^u\right) = \frac{d}{du}\left(5^u\right) \cdot \frac{du}{dx}$

set $u = x^2$

$$= 5^u \cdot \ln(5) \cdot (2x)$$

$$= 5^{x^2} \cdot \ln(5) \cdot (2x)$$

3.5 Implicit differentiation

So far, we have been working with functions

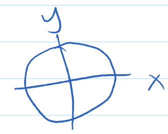
$$\boxed{y = f(x)}$$

↑ a function that depends on x

To differentiate things means to find $\frac{dy}{dx}$

How do you differentiate an equation of the form below

$$\boxed{x^2 + y^2 = 1}$$



(equation of a circle)

Q. How do you find $\frac{dy}{dx}$?

A. we will use the chain rule

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

Aside

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\cdot\frac{dy}{dx}$$

$$= 2y$$

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differentiate $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(6xy)$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\cdot\frac{d}{dx}(y) + y\cdot\frac{d}{dx}(6x)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + y\cdot 6$$

Aside

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3)\cdot\frac{dy}{dx}$$

$$= 3y^2\cdot\frac{dy}{dx}$$

$$\frac{d}{dx}(y) = \frac{d}{dy}(y)\cdot\frac{dy}{dx}$$

$$= 1\cdot\frac{dy}{dx}$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6x \frac{dy}{dx} + y \cdot 6$$

$$3y^2 \frac{dy}{dx} - 6x \frac{dy}{dx} = 6y - 3x^2$$

$$\frac{dy}{dx}\left(3y^2 - 6x\right) = 6y - 3x^2$$

$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{\cancel{3}\left(2y - x^2\right)}{\cancel{3}\left(y^2 - 2x\right)}$$