3.4 The Chain Rule

How do you differentiate a function such as

$$F(x) = \sqrt{x^2+1}$$

All the rules we've seen so far does not help us

Reason - It is a composition of functions

$$F(x) = \sqrt{x^2+1} = (f \circ g)(x) = f(g(x))$$

take $\boxed{f(x) = \sqrt{x}}$, $\boxed{g(x) = x^2+1}$

$$(f \circ g)(x) = f(g(x)) = f(x^2+1) = \sqrt{x^2+1}$$

we know how to differentiate $f(x) = \sqrt{x}$ and $g(x) = x^2+1$

so we need a rule that helps us different $F(x) = (f \circ g)(x)$

The Chain Rule

Differentiation & Integration

FTC

Newton & Leibniz

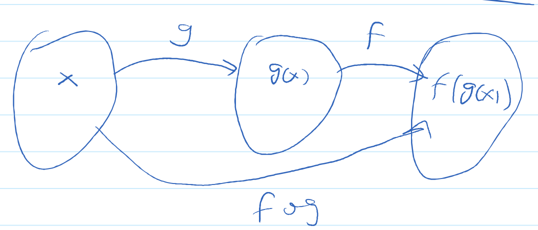
If g is differentiable at x

and f is differentiable at $g(x)$

then $F = f \circ g$ defined by

$$F(x) = f(g(x))$$

Approach 1



is differentiable at x

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

Alternately $y = f(g(x))$

Approach 2

if $\quad y = f(u)$, $u = g(x)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Exercise

Differentiate $\quad F(x) = \sqrt{x^2+1}$

Approach 1

$$y = \sqrt{x^2+1}$$

$$y = \sqrt{u} \quad , \quad u = x^2+1$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot 2x$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

$$y = \sqrt{u} = u^{1/2}$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1}$$

$$= \frac{1}{2}u^{-\frac{1}{2}}$$

$$= \frac{1}{2u^{1/2}}$$

$$= \frac{1}{2\sqrt{u}}$$

$$u = x^2+1$$

$$\frac{du}{dx} = 2x$$

$$F(x) = \sqrt{x^2+1}$$

Find $F'(x)$

Solution

$$F(x) = (f \circ g)(x) = f(g(x))$$

$$f = \sqrt{g(x)} \quad , \quad g(x) = x^2+1$$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2(g(x))^{1/2}} \cdot 2x$$

$$= \frac{1}{2\sqrt{g(x)}} \cdot 2x$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+1}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+1}}$$

$$f = \sqrt{g(x)} = (g(x))^{1/2}$$

$$f'(g(x)) = \frac{1}{2}(g(x))^{-\frac{1}{2}}$$

$$= \frac{1}{2(g(x))^{1/2}}$$

$$g(x) = x^2+1$$

$$g'(x) = 2x$$

$$= \frac{1}{\cancel{x}\sqrt{x^2+1}} \cdot \cancel{x}\,x \qquad = \frac{''}{\sqrt{x^2+1}}$$

Suppose $\quad F(x) = (f \circ g \circ h)(x) = f(g(h(x)))$

Approach 1

$$F'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Approach 2

$y = f(g(h(x)))$ $\qquad w = g(h(x))$, $\quad u = h(x)$

$y = f(w)$ $\qquad w = g(u)$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$x \xrightarrow{h} h(x) \xrightarrow{g} g(h(x)) \xrightarrow{f} f(g(h(x)))$$

$F(x) = \sqrt{x^2+1}$

$\quad = f(g(x))$

$f(g(x)) = \sqrt{g(x)}$, $\quad g(x) = x^2+1$

$$\boxed{F'(x) = f'(g(x)) \cdot g'(x)}$$

outer fn — evaluated at the inner fn — derivative of inner fn

---

Exercise

(1) $y = \sin(x^2)$, differentiate y

$y = \sin(u)$

set $u = x^2$

$y = \sin(u)$ $\qquad u = x^2$

$\frac{dy}{du} = \cos(u)$ $\qquad \frac{du}{dx} = 2x$

$\frac{dy}{dx} = \frac{dy}{du} \quad \frac{du}{}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot 2x$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x \cos(x^2)$$

$\frac{dy}{du} = \cos(u)$ | $\frac{dy}{dx} = 2x$

$$\boxed{\begin{aligned} y &= \sin(x) \\ \frac{dy}{dx} &= \cos(x) \end{aligned}}$$

Exercise

$y = \sin(x^2)$ set $u = x^2$

$y = \sin(u)$ $\frac{du}{dx} = 2x$

$\frac{dy}{du} = \cos(u)$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

---

(2) $y = [\sin(x)]^2$, set $u = \sin(x)$

$y = u^2$

$y = u^2$ $\frac{du}{dx} = \cos(x)$

$\frac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 2u \cdot \cos(x)$$

$$= 2\sin(x) \cdot \cos(x)$$

Power Rule + chain Rule

If n is a real number

$u = g(x)$ is differentiable

$$\frac{d}{dx}(u^n) = n u^{n-1} \cdot \frac{du}{dx}$$

Example

$y = (x^3 - 1)^{100}$, $y = u^{100}$

Example

$y = (x^3-1)^{100}$, $u = x^3 - 1$

$y = u^{100}$ $\quad \frac{du}{dx} = 3x^2$

find $\frac{dy}{dx} = 100\, u^{100-1} \cdot \frac{du}{dx}$

$= 100\, u^{99} \cdot 3x^2$

$= 100\,(x^3-1)^{99} \cdot 3x^2$

$= 300\, x^2 (x^3-1)^{99}$

$y = (x^3-1)^{100}$, $y = u^{100}$

set $u = x^3 - 1$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= 100\, u^{100-1} \cdot 3x^2$

$= 100\, u^{99} \cdot 3x^2$

$= 300\, x^2 (x^3-1)^{99}$

Exercise

$y = \left(\frac{t-2}{2t+1}\right)^9$, find $\frac{dy}{dt}$

set $u = \frac{t-2}{2t+1}$

$y = u^9$, $\frac{dy}{du} = 9u^{9-1}$

$$\boxed{\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}}$$

$= 9u^8 \cdot \frac{5}{(2t+1)^2}$

$= 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{5}{(2t+1)^2}$

$= \frac{9\,(t-2)^8}{(2t+1)^8} \cdot \frac{5}{(2t+1)^2} = \frac{45\,(t-2)^8}{(2t+1)^{10}}$

$\frac{du}{dt} = \frac{(2t+1)\cdot 1 - (t-2)2}{(2t+1)^2}$

$= \frac{\cancel{2t} + 1 - \cancel{2t} + 4}{(2t+1)^2}$

$= \frac{5}{(2t+1)^2}$