Recall the Product Rule and Quotient Rule

Product Rule

$$(fg)' = fg' + gf'$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example

differentiate

$$(2x+3)(x^2+3x+5)$$

$$= (2x+3)\frac{d}{dx}(x^2+3x+5) + (x^2+3x+5)\frac{d}{dx}(2x+3)$$

$$= (2x+3)(2x+3+0) + (x^2+3x+5)(2+0)$$

$$= (2x+3)^2 + 2(x^2+3x+5)$$

$$= 4x^2+12x+9 + 2x^2+6x+10$$

$$= 6x^2+18x+19$$

Rule where $c$ is any real number

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(5) = 0$$

$$\frac{d}{dx}(3) = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}\right) = 0$$

Differentiate

$$f(x) = \frac{2x+3}{x^2+3x+5}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$= \frac{(x^2+3x+5)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(x^2+3x+5)}{(x^2+3x+5)^2}$$

$$= \frac{(x^2+3x+5)(2+0) - (2x+3)(2x+3+0)}{(x^2+3x+5)^2}$$

$$= \frac{2x^2+6x+10 - (4x^2+12x+9)}{(x^2+3x+5)^2}$$

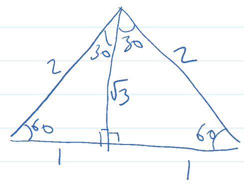
$$= \frac{2x^2+6x+10-4x^2-12x-9}{(x^2+3x+5)^2}$$

$$= \frac{-2x^2-6x+1}{(x^2+3x+5)^2}$$

## 3.3 Derivatives of Trigonometric functions

Brief review of trigonometric function

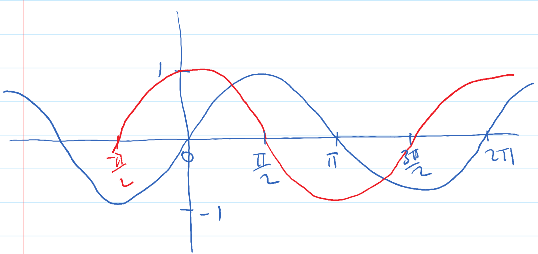
$f(x) = \sin(x)$




$\text{Sine}(\theta) = \frac{\text{opp}}{\text{Hyp}}$

$\text{Cosine}(\theta) = \frac{\text{adj}}{\text{Hyp}}$

$\text{Tangent}(\theta) = \frac{\text{opp}}{\text{adj}}$




radians — degrees

$\sin\left(\frac{\pi}{6}\right) = \sin(30) = \frac{1}{2} = 0.5$

$\sin\left(\frac{\pi}{3}\right) = \sin(60) = \frac{\sqrt{3}}{2} = 0.86\ldots$

$\sin\left(\frac{\pi}{2}\right) = \sin(90) = \frac{2}{2} = 1$

$\cos\left(\frac{\pi}{6}\right) = \cos(30) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \cos(60) = \frac{1}{2}$

$\cos\left(\frac{\pi}{2}\right)$

$$\boxed{\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)}$$

Review Trigonometric function (see pre-calculus)

**Exercise**

Differentiate (we want to show that $f'(x) = \cos(x)$)

$f(x) = \sin(x)$, $f(x+h) = \sin(x+h)$

**Recall Trigonometric identities**

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left[ \frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h} \right]$$

$$= \lim_{h\to 0} \left[ \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\frac{\sin(h)}{h} \right]$$

$$= \lim_{h\to 0}\left( \sin(x)\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\,\frac{\sin(h)}{h}\right)$$

$$= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)\right] + \lim_{h\to 0}\left[\cos(x)\,\frac{\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \quad \text{—— equation } (*)$$

We know

$\lim_{h\to 0}\sin(x) = \sin(x)$ $\qquad \boxed{\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) = 0}$ (prove)

$\lim_{h\to 0}\cos(x) = \cos(x)$ $\qquad \boxed{\lim_{h\to 0}\frac{\sin(h)}{h} = 1}$ (prove)

Equation $(*)$ simplify as follows:

$f(x) = \sin(x)$

$$f'(x) = \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \cos(x)$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

Exercise

use a similar proof to show that

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Summary

1. $\frac{d}{dx}(\sin(x)) = \cos(x)$ (prove)
2. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
3. $\frac{d}{dx}(\tan(x)) = \sec^2(x) = \frac{1}{\cos^2(x)}$
4. $\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = -\csc(x)\cdot\cot(x)$

5. $\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \sec(x)\cdot\tan(x)$

6. $\frac{d}{dx}(\cot(x)) = \frac{d}{dx}\left(\frac{1}{\tan(x)}\right) = -\csc^2(x)$

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Recall

If $f(x) = \sin(x)$ , $f'(x) = \cos(x)$

Exercise

Prove that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$

Pf

We will use the quotient Rule

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\cdot\frac{d}{dx}(\sin(x)) - \sin(x)\frac{d}{dx}(\cos(x))}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos(x)\cdot\cos(x) + \sin(x)\sin(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)}$$

$$= \left(\frac{1}{\cos(x)}\right)^2$$

$$\boxed{\frac{d}{dx}(\tan(x)) = \sec^2(x)}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$\text{differentiate}\left(\frac{f}{g}\right) = \frac{g\frac{d}{dx}(f) - f\frac{d}{dx}(g)}{g^2}$$

Recall

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

$\frac{d}{dx}(\sin(x)) = \cos(x)$ ✓

$\frac{d}{dx}(\cos(x)) = -\sin(x)$

Recall

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

If $A = B$

$\cos(A-A) = \cos A\cos A + \sin A\sin A$

$\cos(0) = \cos^2 A + \sin^2 A$

$1 = \cos^2 A + \sin^2 A$

Recall

$\sec(x) = \frac{1}{\cos(x)}$

---

Recall in the proof for $\frac{d}{dx}(\sin(x)) = \cos(x)$

we used

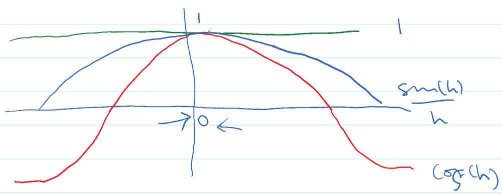
$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

Prove $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$

Pf (hand waving proof)

verify with a graphing calculator that (desmos)

$$\boxed{\cos(h) \le \frac{\sin(h)}{h} \le 1}$$




$\lim_{h \to 0} \cos(h) = 1$ ✓

$\lim_{h \to 0} 1 = 1$ ✓

By the Squeeze theorem

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

Recall the Squeeze theorem

if $f(x) \le g(x) \le h(x)$

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

then $\lim_{x \to a} g(x) = L$

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# Quiz

Using $\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$, show that

1. if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$ ✓

2. If $f(x) = \frac{1-x}{2+x}$ then $f'(x) = \frac{-3}{(2+x)^2}$ ✓

---

Corrections

using $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

1. show that if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$

$f(x+h) = \sqrt{x+h}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$\sqrt{x} \cdot \sqrt{x} = x$

$x^{1/2} \cdot x^{1/2} = x^{\frac{1}{2}+\frac{1}{2}}$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$\sqrt{x} \cdot \sqrt{x} = x$

$x^{1/2} \cdot x^{1/2} = x^{\frac{1}{2}+\frac{1}{2}}$

$= x$

$$= \lim_{h \to 0} \frac{x + h + \cancel{\sqrt{x}\sqrt{x+h}} - \cancel{\sqrt{x}\sqrt{x+h}} - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h \cancel{-x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

2) If $f(x) = \frac{1-x}{2+x}$ then $f'(x) = \frac{-3}{(2+x)^2}$

$f(x) = \frac{1-x}{2+x}$

$f(x+h) = \frac{1-(x+h)}{2+(x+h)}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h}$$

$$\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}$$

$$\frac{(2+x)(1-x-h) - (2+x+h)(1-x)}{(2+x)(2+x+h)}$$

$$= \lim_{h \to 0} \frac{(2+x)(1-x-h) - (2+x+h)(1-x)}{h(2+x)(2+x+h)}$$

$$= \lim_{h \to 0} \frac{2 - 2x - 2h + x - x^2 - xh - (2 - 2x + x - x^2 + h - xh)}{h(2+x)(2+x+h)}$$

$$= \lim_{h \to 0} \frac{\cancel{2} - \cancel{2x} - 2h + \cancel{x} - \cancel{x^2} - \cancel{xh} - \cancel{2} + \cancel{2x} - \cancel{x} + \cancel{x^2} - h + \cancel{xh}}{h(2+x)(2+x+h)}$$

$$= \lim_{h \to 0} \frac{-3\cancel{h}}{\cancel{h}(2+x)(2+x+h)}$$

$$= \frac{-3}{(2+x)(2+x)} = \frac{-3}{(2+x)^2}$$

$$= \frac{-3}{(2+x)(2+x)} = \frac{-3}{(2+x)^2}$$