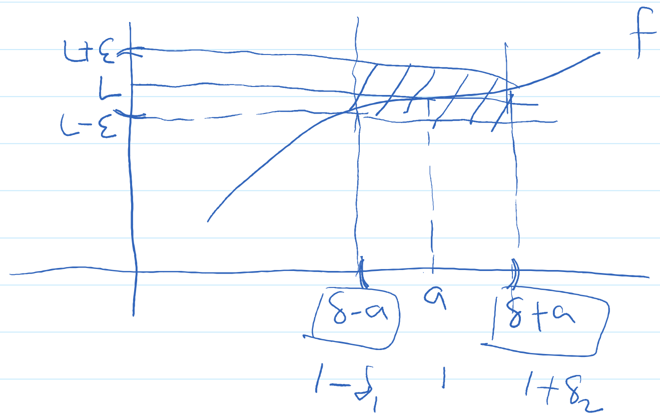
2.4 Precise definition of the limit

given an $\boxed{\varepsilon > 0}$, there exist a $\boxed{\delta > 0}$ such that

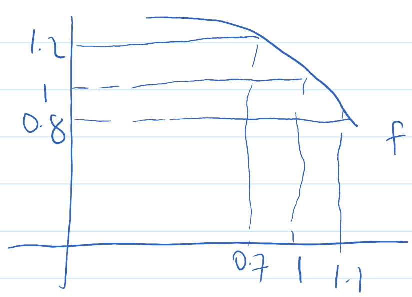
If $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$



Exercises.

#1. Use given graph

find δ such that if $|x-1| < \delta$ then $|f(x) - 1| < 0.2$



$|f(x) - 1| < 0.2$

$-0.2 < f(x) - 1 < 0.2$

$-0.2 + 1 < f(x) - 1 + 1 < 0.2 + 1$

$0.8 < f(x) < 1.2$

We see from the graph

$0.8 < f(x) < 1.2$ is true

if

$0.7 < x < 1.1$

$1 - \delta_1 = 0.7$

$\delta_1 = 1 - 0.7 = 0.3$

$1 + \delta_2 = 1.1$

$\delta_2 = 1.1 - 1 = 0.1$

$\delta = \min\{0.1, 0.3\} = \boxed{0.1}$

or anything smaller.

#2.

If $|x-2|<\delta$, then $|\sqrt{x^2+5}-3|<0.4$

$|\sqrt{x^2+5}-3|<0.4$

$-0.4<\sqrt{x^2+5}-3<0.4$

$-0.4+3<\sqrt{x^2+5}<0.4+3$

$2.6<\sqrt{x^2+5}<3.4$

solve for x when

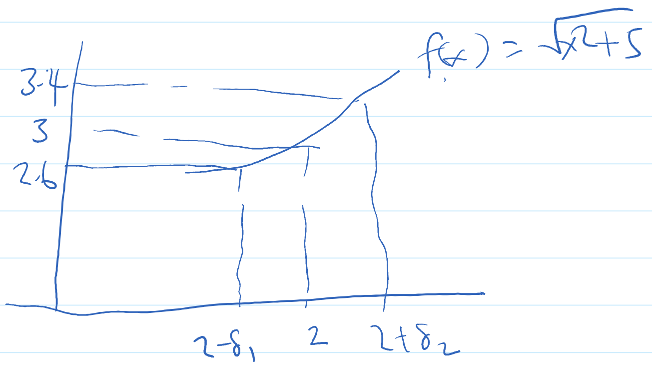
$y=\sqrt{x^2+5}=2.6$

$x^2+5=2.6^2$

$x^2=2.6^2-5$

$x=\sqrt{2.6^2-5}$

$x=\sqrt{1.76}\approx 1.3266$



If $|x-2|<\delta$

then $|\sqrt{x^2+5}-3|<0.2$

$2-\delta_1\approx 1.3266$

$\delta_1\approx 2-1.3266=0.6734$

solve for x

$y=\sqrt{x^2+5}=3.4$

$x^2+5=3.4^2$

$x^2=3.4^2-5$

$x=\sqrt{3.4^2-5}$

$=\sqrt{6.56}$

$=2.5612$

$2+\delta_2\approx 2.5612$

$\delta_2=2.5612-2$

$=0.5612$

$\delta=\min\{\delta_1,\delta_2\}$

$=\min\{\underline{0.5612}, 0.6734\}$

#3. find δ such that

(a) If $|x-1|<\delta$ then $|5x-5|<\varepsilon$, where $\varepsilon=0.5$

$$|5x-5| = |5(x-1)|$$
$$= 5|x-1| < \varepsilon$$
$$\Rightarrow \quad |x-1| < \frac{\varepsilon}{5} \qquad (\varepsilon = 0.5)$$
$$|x-1| < \frac{0.5}{5}$$
$$|x-1| < 0.1 \qquad \text{choose } \delta = \boxed{0.1}$$

(5) repeat (4) for $\varepsilon = 0.05$

$$|5x-5| = 5|x-1| < \varepsilon$$
$$|x-1| < \frac{\varepsilon}{5} \qquad (\varepsilon = 0.05)$$
$$|x-1| < \frac{0.05}{5}$$
$$|x-1| < \boxed{0.01}$$

If $|x-1| < \delta$ then $|5x-5| < \varepsilon$ $(\varepsilon = 0.05)$

choose $\delta = \underline{0.01}$.

#6. use ε–δ definition

$$\lim_{x\to 4}\left(\frac{1}{4}x+1\right) = 2 \qquad \boxed{\text{given } \varepsilon > 0}$$

find $\delta > 0$

If $0 < |x-4| < \delta$ then $\left|\left(\frac{1}{4}x+1\right) - 2\right| < \varepsilon$

$$\left| \frac{1}{4}x + 1 - 2 \right| = \left| \frac{1}{4}x - 1 \right| < \varepsilon$$

$$\Rightarrow \frac{1}{4}\left| x - 4 \right| < \varepsilon$$

$$\Rightarrow |x - 4| < 4\varepsilon$$

choose $\delta = 4\varepsilon$

#5 use $\varepsilon - \delta$ definition of the limit

$$\lim_{x \to 2} (x^2 - 4x + 9) = 5$$

$\boxed{\text{given } \varepsilon > 0}$ ✓ find $\delta > 0$

pf.

If $0 < |x-2| < \delta$ then $\left| (x^2 - 4x + 9) - 5 \right| < \varepsilon$

$$|x^2 - 4x + 9 - 5| = |x^2 - 4x + 4|$$

$$= |(x-2)^2| < \varepsilon$$

$(x-2)^2$
$= (x-2)(x-2)$
$x^2 - 2x - 2x + 4$

$$= |x-2|^2 < \varepsilon$$

$$\Rightarrow |x-2| < \sqrt{\varepsilon}$$

choose $\delta = \sqrt{\varepsilon}$

#6. H - Heaviside function

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Show that $\lim_{t \to 0} H(t)$ DNE

If $0 < |t-0| < \delta$, then $|H(t) - L| < \varepsilon$

Pf by contradiction (pick a particular $\varepsilon = \frac{1}{2}$

given $\varepsilon = \frac{1}{2}$ then there exist $\delta > 0$ such that
$0 < |t| < \delta$

If $0 < |t| < \delta$ then $|H(t) - L| < \frac{1}{2}$

$0 < t < \delta$

$-\delta < t < 0$

$$-\frac{1}{2} < H(t) - L < \frac{1}{2}$$

$$L - \frac{1}{2} < H(t) < L + \frac{1}{2}$$

$0 < t < \delta$, $H(t) = 1$, $1 < L + \frac{1}{2}$

$L > 1 - \frac{1}{2}$

$\boxed{L > \frac{1}{2}}$

for $-\delta < t < 0$, $H(t) = 0$ $L - \frac{1}{2} < 0$

$\boxed{L < \frac{1}{2}}$

Contradiction

~~Basi~~ Therefore $\lim_{t \to 0} H(t)$ DNE

#7. given $\varepsilon > 0$ find $\delta > 0$ such that if $0 < |x-4| < \delta$
then $|(5x-10) - 10| < \varepsilon$

$$|5x-10-10| = |5x-20| = |5(x-4)|$$

$$= 5|x-4| < \varepsilon$$

$$\Rightarrow \quad |x-4| < \frac{\varepsilon}{5}$$

choose $\delta = \varepsilon/5$

check if your choice for δ is good $\left(\delta = \frac{\varepsilon}{5}\right)$

if $0 < |x-4| < \delta$

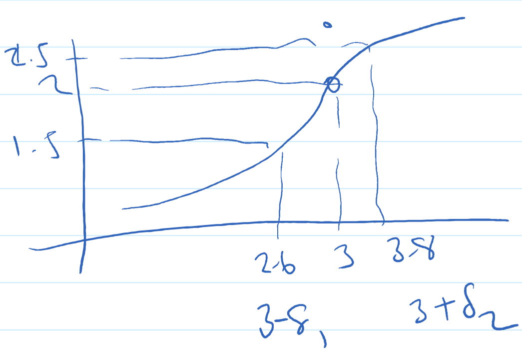
$$|(5x-10)-10| = |5x-20|$$

$$= 5|x-4|$$

$$< 5\delta = 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

#8. Use the graph to find a δ such that

if $0 < |x-3| < \delta$ then $|f(x)-2| < 0.5$



$$|f(x)-2| < 0.5$$

$$-0.5 < f(x)-2 < 0.5$$

$$-0.5+2 < f(x) < 0.5+2$$

$$1.5 < f(x) < 2.5$$

from the graph

$1.5 < f(x) < 2.5$ corresponds to

$$2.6 < x < 3.8$$

To choose δ_1, $\quad 3-\delta_1 = 2.6$

To choose δ_1, $3-\delta_1 = 2.6$

$\delta_1 = 3-2.6 = 0.4$

To choose δ_2, $3+\delta_2 = 3.8$

$\delta_2 = 3.8-3 = 0.8$

$$\delta = \min\{\delta_1, \delta_2\} = \min\{0.4, 0.8\} = \underline{\underline{0.4}}$$

#9. Do

#10. $\lim_{x \to 2} (x^3 - 5x + 9) = 7$

(a) $\varepsilon = 0.2$, find δ

given $\varepsilon = 0.2$, find δ such that

If $0 < |x-2| < \delta$ then $|(x^3-5x+9)-7| < 0.2$

$|x^3-5x+9-7| < 0.2$

$-0.2 < x^3-5x+2 < 0.2$

find x when $y = x^3-5x+2 = -0.2$

$y = x^3-5x+2.2$

Using a graphing calculator

$x = 1.970$

$2-\delta_1 = 1.970$

choose $\delta_1 = 2-1.970 = 0.03$

when $y = x^3 - 5x + 9 - 7 = 0.2$

$y = x^3 - 5x + 2 - 0.2$

$y = x^3 - 5x + 1.8$

use a graphing calculator

to x

$x \approx 2.0279$

$\delta_2 + 2 = 2.0279$

$\delta_2 = 0.0279$

$\delta = \min\{\delta_1, \delta_2\} = \min\{0.03, 0.0279\}$

$= \underline{\underline{0.0279}}$ ✓