Recall the Product Rule and Quotient Rule

Product Rule

$$(fg)' = fg' + gf'$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Example

differentiate

$$(2x+3)(x^2+3x+5)$$

$$= (2x+3)\frac{d}{dx}(x^2+3x+5) + (x^2+3x+5)\frac{d}{dx}(2x+3)$$

$$= (2x+3)(2x+3+0) + (x^2+3x+5)(2+0)$$

$$= (2x+3)^2 + 2(x^2+3x+5)$$

$$= 4x^2+12x+9 + 2x^2+6x+10$$

$$= 6x^2+18x+19$$

Rule where c is any real number

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(5) = 0$$

$$\frac{d}{dx}(3) = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}\right) = 0$$

Differentiate

$$f(x) = \frac{2x+3}{x^2+3x+5}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$= \frac{(x^2+3x+5)\frac{d}{dx}(2x+3) - (2x+3)\frac{d}{dx}(x^2+3x+5)}{(x^2+3x+5)^2}$$

$$= \frac{(x^2+3x+5)(2+0) - (2x+3)(2x+3+0)}{(x^2+3x+5)^2}$$

$$= \frac{2x^2+6x+10 - (4x^2+12x+9)}{(x^2+3x+5)^2}$$

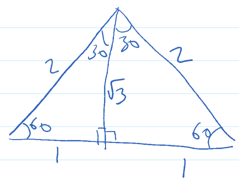
$$= \frac{2x^2+6x+10-4x^2-12x-9}{(x^2+3x+5)^2}$$

$$= \frac{-2x^2-6x+1}{(x^2+3x+5)^2}$$

3.3 Derivatives of Trigonometric functions

Brief review of trigonometric function

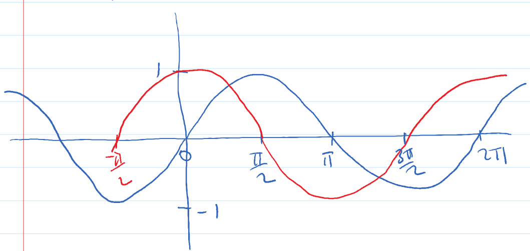
$f(x) = \sin(x)$



$\text{Sine}(\theta) = \frac{\text{opp}}{\text{Hyp}}$

$\text{Cosine}(\theta) = \frac{\text{adj}}{\text{Hyp}}$

$\text{Tangent}(\theta) = \frac{\text{opp}}{\text{adj}}$



radians   degrees

$\sin\left(\frac{\pi}{6}\right) = \sin(30) = \frac{1}{2} = 0.5$

$\sin\left(\frac{\pi}{3}\right) = \sin(60) = \frac{\sqrt{3}}{2} = 0.86\ldots$

$\sin\left(\frac{\pi}{2}\right) = \sin(90) = \frac{2}{2} = 1$

$\cos\left(\frac{\pi}{6}\right) = \cos(30) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \cos(60) = \frac{1}{2}$

$\cos\left(\frac{\pi}{2}\right)$

$$\boxed{\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)}$$

Review Trigonometric function (see pre-calculus)

Exercise

Differentiate

$f(x) = \sin(x)$, $f(x+h) = \sin(x+h)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}\right]$$

$$= \lim_{h\to 0} \left[\sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\,\frac{\sin(h)}{h}\right]$$

$$= \lim \left[\sin(x)\left(\frac{\cos(h) - 1}{}\right)\right] + \lim \left[\cos(x)\,\frac{\sin(h)}{}\right]$$

Recall Trigonometric identities

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$= \lim_{h\to 0}\left[\sin(x)\left(\frac{\cos(h)-1}{h}\right)\right] + \lim_{h\to 0}\left[\cos(x)\ \frac{\sin(h)}{h}\right]$$

$$= \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h} \quad \text{— equation } (*)$$

We know

$\lim_{h\to 0}\sin(x) = \sin(x)$ $\qquad \lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) = 0$ (prove)

$\lim_{h\to 0}\cos(x) = \cos(x)$ $\qquad \lim_{h\to 0}\frac{\sin(h)}{h} = 1$ (prove)

Equation $(*)$ simplify as follows:

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{h\to 0}\sin(x)\cdot\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0}\cos(x)\cdot\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1$$

$$= \cos(x)$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

Exercise

use a similar proof to show that

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

Summary

1. $\frac{d}{dx}(\sin(x)) = \cos(x)$ (Prove)
2. $\frac{d}{dx}(\cos(x)) = -\sin(x)$
3. $\frac{d}{dx}(\tan(x)) = \sec^2(x) = \frac{1}{\cos^2(x)}$
4. $\frac{d}{dx}(\csc(x)) = \frac{d}{dx}\left(\frac{1}{\sin(x)}\right) = -\csc(x)\cdot\cot(x)$
5. $\frac{d}{dx}(\sec(x)) = \frac{d}{dx}\left(\frac{1}{\cos(x)}\right) = \sec(x)\cdot\tan(x)$

6. $\frac{d}{dx}\left(\cot(x)\right) = \frac{d}{dx}\left(\frac{1}{\tan(x)}\right) = -\csc^2(x)$