2.3 Calculating Limits Using the Limit Laws

Recap from 2.2 (limit of function)

Remark

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \text{ (left-hand limit)}$$

and

equals

$$\lim_{x\to a^+} f(x) = L \text{ (Right-hand limit)}$$

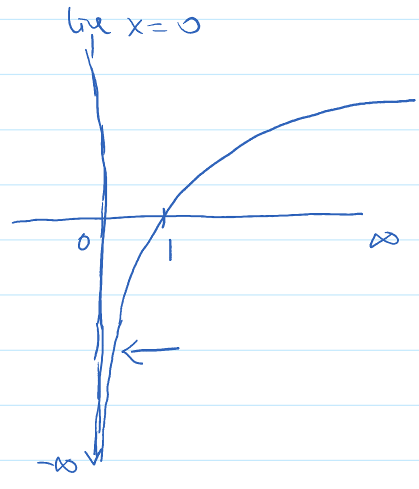
Definition

Infinite limits and Vertical Asymptotes

$$\lim_{x\to a} f(x) = \infty \implies$$ line $x = A$ is a vertical asymptote to the curve $f(x)$

Example

$$\lim_{x\to 0^+} \ln x = -\infty$$



$\implies$ line $x = 0$ is a vertical asymptote to the curve $\ln x$

Limit Laws

Suppose that 'c' is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist, then

① $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

② $\lim [f(x) - g(x)] = \lim f(x) - \lim g(x)$

$x \to a$

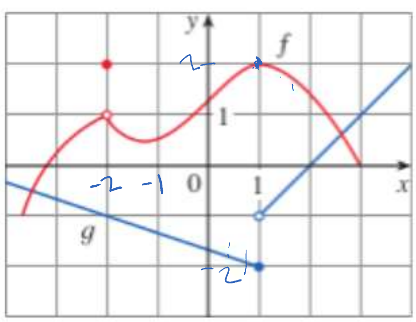
② $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$

③ $\lim\limits_{x \to a} [f(x)\, g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

④ $\lim\limits_{x \to a} [c \cdot f(x)] = c \cdot \lim\limits_{x \to a} f(x)$

⑤ $\lim\limits_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, where $\left(\lim\limits_{x \to a} g(x) \neq 0\right)$

Exercise



f in 'red'

g in 'blue'

ⓐ $\lim\limits_{x \to 2} [f(x) + 5g(x)]$

$= \lim\limits_{x \to 2} f(x) + \lim\limits_{x \to 2} (5 \cdot g(x))$

$= \lim\limits_{x \to 2} f(x) + 5 \lim\limits_{x \to 2} g(x)$

$= 1 + 5(-1)$

$= 1 - 5 = -4$

ⓑ $\lim\limits_{x \to 1} [f(x) \cdot g(x)]$

$= \lim\limits_{x \to 1} f(x) \cdot \lim\limits_{x \to 1} g(x)$

left hand limit

$\lim\limits_{x \to 1^-} f(x) \cdot \lim\limits_{x \to 1^-} g(x)$

$2 \cdot (-2) = -4$

right hand limit

$\lim\limits_{x \to 1^+} f(x) \cdot \lim\limits_{x \to 1^+} g(x)$

$2 \cdot (-1) = -2$

$\lim\limits_{x \to 1^-} [f(x) \cdot g(x)] = -4$

not equal to

$\lim\limits_{x \to 1^+} [f(x) \cdot g(x)] = -2$

So $\lim [f(x) \cdot g(x)]$ DNE

So $\lim_{x \to 1} [f(x) \cdot g(x)]$ DNE

Power law

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \qquad (n > 0)$$

Root law

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\left[\lim_{x \to a} f(x)\right]} \qquad (n > 0)$$

$$\lim_{x \to a} (f(x))^{1/n} = \left[\lim_{x \to a} f(x)\right]^{1/n}$$

(when n is even, $\lim_{x \to a} f(x) > 0$)

when we limit ourselves to Real

$\sqrt{-8}$ (error) ✗

$\sqrt[3]{-8}$ (possible) ✓

Special limits

① $\lim_{x \to a} c = c$

② $\lim_{x \to a} x = a$

③ $\lim_{x \to a} x^n = a^n \qquad (n > 0)$

④ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \qquad (n > 0)$

(n is even, $a > 0$)

Exercise. $f(x)$ $g(x)$

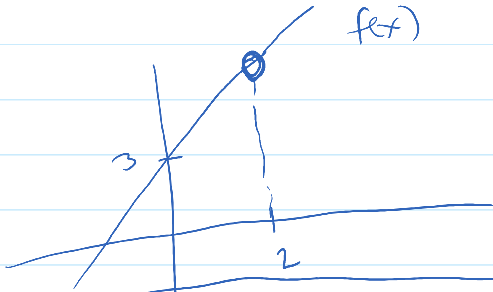
$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

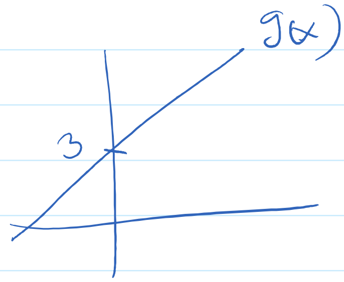
Dom f

$(-\infty,2)\cup(2,\infty)$

$$\frac{(x-2)(x+3)}{(x-2)} = x+3$$

'naive'





$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2} (x+3)$$

$$= 2+3 = 5$$

$$\lim_{x\to 3} g(x) = \lim_{x\to 2} (x+3)$$

$$= 2+3 = 5$$

If $f(x) = g(x)$ $(x \neq 2)$

then $\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x)$

In general

If $f(x) = g(x)$, $(x \neq a)$

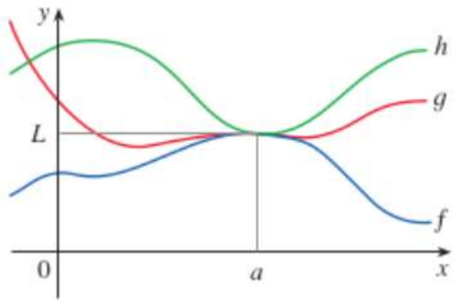
then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

Theorem 1

$\lim_{x\to a} f(x) = L$ If and only if $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$

SQUEEZE THEOREM

Theorem 2



Theorem 2

If $f(x) \le g(x)$ when x is near 'a'

and limit of f and g both exist as x approach 'a'

then

$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$$

Theorem 3 (Sandwich Theorem)

If $f(x) \le g(x) \le h(x)$ when x is near 'a'

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then $\lim_{x \to a} g(x) = L$

Exercise

use squeeze theorem

If $4x - 9 \le f(x) \le x^2 - 4x + 7 \quad (x \ge 0)$

find $\lim_{x \to 4} f(x)$

$$\lim_{x \to 4} 4x - 9 = 4(4) - 9 = 16 - 9 = 7$$

$$\lim_{x \to 4} x^2 - 4(x) + 7 = 4^2 - 4(4) + 7$$
$$= 16 - 16 + 7 = 7$$

$$\lim_{x \to 4} f(x) = 7$$

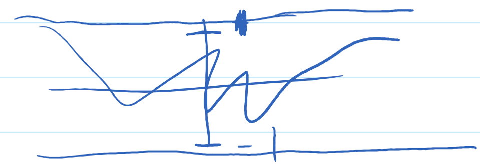
Exercise

Exercise

use squeeze theorem to show that $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$

(we cannot use the product law $\lim_{x \to 0} \sin\frac{1}{x}$ DNE)

graph $\sin\frac{1}{x}$



you will discover that

$$-1 \le \sin\frac{1}{x} \le 1$$

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\lim_{x \to 0} (-x^2) = 0$$

$$\lim_{x \to 0} x^2 = 0$$

by squeeze thm

$$\boxed{\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0}$$