Exam 2, October 7 (the whole day, you will have to find ~~a~~ free 3 hrs to take the exam)

Sections on Exam 2

2.5, 2.6, 2.7, 2.8, 3.1, 3.2, 3.3

New set of homework today (3.1, 3.2, 3.3)

Exam 2 practice today

3 - Differentiation Rules

3.1 Derivatives of polynomials and Exponential function

3.2 Product and Quotient Rules

Recall

f, g are differentiable functions

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

c is a constant (example $c = 5, \pi, \ldots$)

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(x^2) = 2x$

4. $\frac{d}{dx}(x^3) = 3x^2$

$\vdots$

Power rule

5. $\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$

6. $\frac{d}{dx}(c\, f(x)) = c \frac{d}{dx}(f(x))$

example

$$\frac{d}{dx}(3x^2) = 3 \frac{d}{dx}(x^2)$$
$$= 3 \cdot (2x^{2-1})$$
$$= 3 \cdot 2x$$
$$= 6x$$

7. $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

example

5. $\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$

example

$$\frac{d}{dx}(3x^2 + 2x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(2x)$$
$$= 3\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x)$$
$$= 3(2x^{2-1}) + 2(1)$$
$$= 6x + 2$$

8. $\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$

9. $\boxed{\frac{d}{dx}(e^x) = e^x}$ (where $\boxed{e}$ is the Euler number) (Leonhard Euler)

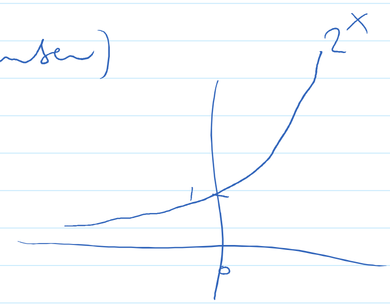
give motivation on why e is important

let

$f(x) = b^x$ (here b is a real number)

$f(x+h) = b^{x+h}$

example $f(x) = 2^x$



find $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x b^h - b^x}{h}$$

$$= \lim_{h \to 0} b^x \frac{(b^h - 1)}{h}$$

$$f'(x) = b^x \boxed{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

$f'(0)$

Recall

$a^{m+n} = a^m \cdot a^n$

$\boxed{f(x) = b^x}$

(The derivative of an exponential function is proportional to the exponential function)

$\boxed{f'(x) = b^x f'(0)}$

$$\boxed{f'(x) = b^x f'(0)}$$

Summary

If $f(x) = b^x$ (b is a real number)

then $\boxed{f'(x) = b^x f'(0)}$ $\left(\frac{d}{dx}(b^x) \propto b^x\right)$

$$f'(0) = \lim_{h\to 0} \frac{b^h - 1}{h}$$

take $b = 2$ or 3

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$	$\frac{e^h - 1}{h}$
0.1	0.71773	1.16123	
0.01	0.69556	1.10467	
0.001	0.69739	1.09922	
0.0001	0.69317	1.09867	$\downarrow$ 1

$$\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$

if $b = 2$ $f'(0) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx 0.693$

$b = 3$ $f'(0) = \lim_{h\to 0} \frac{3^h - 1}{h} \approx 1.099$

(1) If $f(x) = 2^x$

then $f'(x) \approx 0.693\, 2^x$

(2) If $f(x) = 3^x$

then $f'(x) \approx 1.099\, 3^x$

Definition of e

$$\boxed{\lim_{h\to 0} \frac{e^h - 1}{h} = 1}$$

(3) If $f(x) = e^x$

then $f'(x) = 1 \cdot e^x = e^x$

$(f'(0) = 1)$

$\frac{d}{dx}(e^x) = e^x$ $\qquad$ ($e^1 = 2.718....$)
(Irrational number)

Exercise.

If $f(x) = e^x - x$ (a) find $f'(x)$ (b) find $f''(x)$

(a) $\frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x)$

$= e^x - 1$

$f'(x) = e^x - 1$

(b) $\frac{d}{dx}(f'(x)) = \frac{d}{dx}(e^x - 1) = \frac{d}{dx}(e^x) - \frac{d}{dx}(1)$

$= e^x - 0 = e^x$

$f''(x) = e^x$

3.2 Product and Quotient Rule

Product Rule

If f, g are both differentiable

$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$ (Leibniz)

$(fg)' = fg' + gf'$ (Newton)

example

(1) $f(x) = x^2(x+3)$

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}\left[x^2(x+3)\right] = x^2\frac{d}{dx}(x+3) + (x+3)\frac{d}{dx}(x^2)$$
$$= x^2(1+0) + (x+3)(2x)$$
$$= x^2 + 2x(x+3)$$

(2) $f(x) = xe^x$, find $f'(x)$

$$\frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x)$$
$$= xe^x + e^x \cdot 1$$
$$= e^x(x+1)$$

Exercise

If $f(x) = xe^x$, show that $f''(x) = e^x(x+2)$

$f'''(x) = e^x(x+3)$

$\vdots$

(nth derivative) $f^{(n)}(x) = e^x(x+n)$ ✓

(Induction proof)