Chapter 3: Differentiation Rules

Recall

If f is a function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

3.1 Derivatives of Polynomials and Exponential function

Constant functions

$f(x) = c$ (where c is a constant)

example

$f(x) = 5$, $f'(x) = 0$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x) = c$

$f(x+h) = c$

$f(x) = 5$

$$= \lim_{h\to 0} \frac{c - c}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

Rule 1

$$\boxed{\frac{d}{dx}(c) = 0}$$

Leibniz | Newton

example

$\frac{d}{dx}(5) = 0$

if $f(x) = 5$ then $f'(x) = 0$

Rule 2

$$\boxed{\frac{d}{dx}(x) = 1}$$

Prove Rule 2

Pf $f(x) = x$, $f(x+h) = x+h$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

Exercise

find $\frac{d}{dx}(x^2) = 2x$

$f(x) = x^2$, $f(x+h) = (x+h)^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= 2x$$

$\frac{d}{dx}(3x^2) = 6x$

If I can convince you about the power rule

$$\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$$

$$\frac{d}{dx}(3x^2) = 3\cdot 2 x^{2-1}$$

$$= 6x$$

$f(x) = 3x^2$

$f(x+h) = 3(x+h)^2$

$= 3(x^2+2xh+h^2)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2+2xh+h^2)-3x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2}+6xh+3h^2-\cancel{3x^2}}{h}$$

$$= \lim_{h\to 0} \cancel{h}(6x+3h)$$

$$= 6x$$

Power Rule

if n is a positive number

then $\frac{d}{dx}(x^n) = n x^{n-1}$

(If $f(x) = x^n$

then $f'(x) = n x^{n-1}$

$\frac{d}{dx}(x^n) = nx$ ... $f(x) = nx$

Pf.

$f(x) = x^n$, $f(x+h) = (x+h)^n$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{x^n} + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n] - \cancel{x^n}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}]}{\cancel{h}}$$

$$= nx^{n-1}$$

$$\boxed{\frac{d}{dx}(x^n) = nx^{n-1}, \quad n>0}$$

⋮ Hand waving

It is not difficult to extend the result to any n (real number)

$$\boxed{\text{For any real number } n \quad \frac{d}{dx}(x^n) = nx^{n-1}}$$

Binomial theorem

if n is positive

$$(x+h)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + {}^nC_n h^n$$

$${}^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$$

$${}^nC_1 = \frac{n!}{(n-1)!\,1!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

$${}^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2} = \frac{n(n-1)}{2}$$

$${}^nC_{n-1} = \frac{n!}{(n-(n-1))!\,(n-1)!} = \frac{n!}{1!\,(n-1)!} = \frac{n\cancel{(n-1)!}}{\cancel{(n-1)!}} = n$$

$${}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = \frac{n!}{n!} = 1$$

So the Binomial thm becomes

$$(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

Exercise

differentiate

(1) $f(x) = \frac{1}{x^2} = x^{-2}$

Recall

$\sqrt{x} = x^{\frac{1}{2}}$

(a) $f(x) = \frac{1}{x^2} = x^{-2}$

$$\frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3}$$

$$= \frac{-2}{x^3}$$

$\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{1/3}$

$\vdots$

$\sqrt[n]{x} = x^{1/n}$

$(x^m)^{1/n} = x^{\frac{m}{n}}$

(b) $f(x) = \sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$

$$\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3}$$

$$= \frac{2}{3x^{1/3}}$$

$$= \frac{2}{3\sqrt[3]{x}}$$

$\frac{2}{3} - \frac{1 \cdot 3}{3} = \frac{2}{3} - \frac{3}{3}$

$= \frac{-1}{3}$

Constant multiplication Rule

$$\frac{d}{dx}[c\,f(x)] = c\,\frac{d}{dx}[f(x)]$$

Pf

let $g(x) = c\,f(x)$, $g(x+h) = c\,f(x+h)$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{c\,f(x+h) - c\,f(x)}{h}$$

$$= \lim_{h \to 0} c\left(\frac{f(x+h) - f(x)}{h}\right) \quad \text{(using limits laws)}$$

$$= c \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$g'(x) = c\,f'(x)$$

$$\frac{d}{dx}\left(c\, f(x)\right) = c\, \frac{d}{dx}\left[f(x)\right)$$

Exercises

$$\frac{d}{dx}(3x^4) = 3\,\frac{d}{dx}(x^4)$$
$$= 3\left[4 \cdot x^{4-1}\right] = 12x^3$$

Sum and difference rules

If f, g are both differentiable

then $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

$$\frac{d}{dx}\left[f(x) - g(x)\right] = \frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[g(x)\right]$$

Pf (simple, use the definition)

Exercise

$f(x) = x^3 - x$

$$\frac{d}{dx}\left[x^3 - x\right) = \frac{d}{dx}\left[x^3\right) - \frac{d}{dx}(x^1)$$
$$= 3x^{3-1} - 1x^{1-1}$$
$$= 3x^2 - 1 \cdot x^0$$
$$= 3x^2 - 1$$

Recall

$x^0 = 1$

Factorial

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! = 5 \cdot 4! = 5 \cdot 4 \cdot 3! = 5 \cdot 4 \cdot 3 \cdot 2!$$

$$4 \cdot 3 \cdot 2 \cdot 1 = 4! = 4 \cdot 3! = 4 \cdot 3 \cdot 2! = 4 \cdot 3 \cdot 2 \cdot 1!$$

$$3 \cdot 2 \cdot 1 = 3! = 3 \cdot 2! = 3 \cdot 2 \cdot 1!$$

$$2 \cdot 1 = 2! = 2 \cdot 1!$$

$$1 \cdot 1 = 1!$$

$$\boxed{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot} 4 \cdot 3 \cdot 2 \cdot 1 = 10!$$