Tuesday, September 22, 2020 8:00 AM

$\boxed{2.7}$ Derivatives and Rates of Change

Definition

The Derivative of a function f at a point a is denoted $f'(a)$

(i) $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ ✓

(ii) $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ ✓

Exercise

use definition (i) to find the derivative of

$f(x) = x^2 - 8x + 9$

(a) at 2

$f(2) = 2^2 - 8(2) + 9$

$f(2+h) = (2+h)^2 - 8(2+h) + 9$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (2^2 - 8(2) + 9)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{16} - 8h \cancel{+9} \cancel{-4} \cancel{+16} \cancel{-9}}{h}$$

$$= \lim_{h \to 0} \frac{-4h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-4 + h)}{\cancel{h}}$$

$$= \lim_{h \to 0} -4 + h$$

$$= \boxed{-4}$$

(b) ...

(b) $f'(a)$ Exercise

In summary, 2.7:

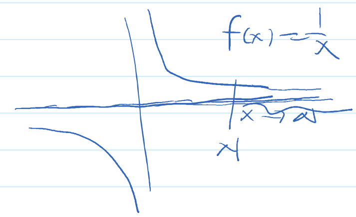
Remark

The tangent line to the curve $y=f(x)$ at $(a, f(a))$ is line passing through $(a, f(a))$ whose slope is $\boxed{f'(a)}$ which is the derivative of f at 'a'

Exercise.

use $(\varepsilon - N)$ definition to prove

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



Solution

given $\boxed{\varepsilon > 0}$, find N such that

if $x > N$ then $|f(x) - L| < \varepsilon$

$$\left|\frac{1}{x} - 0\right| < \varepsilon$$

$$\left|\frac{1}{x}\right| < \varepsilon$$

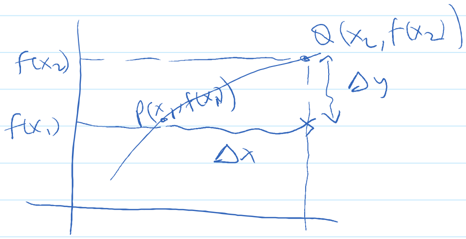
$$\frac{1}{x} < \varepsilon$$

$$x > \boxed{\frac{1}{\varepsilon}}$$

choose $N = \frac{1}{\varepsilon}$

Rates of change

$Q(x_2, f(x_2))$



Instantaneous rate of change $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ ✓

2.8 The derivative as a function

Previously (derivative of function f at a point a)

Derivative at a point 'a' $\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

Now suppose we let a vary

(replace a with x)

Imp derivative at every point x in the domain of f

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

given any x in the Domain of f and as long as this limit exist

we say $\boxed{f'}$ is the derivative of $\boxed{f}$

$\uparrow$ function $\qquad \uparrow$ function

Remark

at every point on the graph of f we can draw a tangent line to construct the graph of f'

Exercise

If $f(x) = x^3 - x$, find $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x+h) = (x+h)^3 - (x+h)$

$f(x) = x^3 - x$

$$h \to 0 \qquad h \qquad\qquad f(x) = x^3 - x$$

$$= \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - (x^3 - x)}{h}$$

$$= \lim_{h\to 0} \frac{[\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}]}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

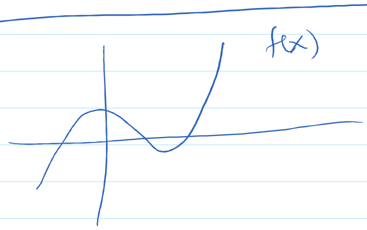
$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 - 1)}{\cancel{h}}$$

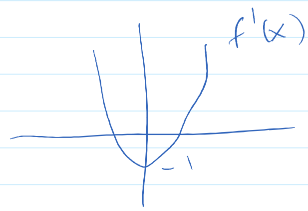
$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 - 1$$

$$= 3x^2 + \cancel{3x(0)} + \cancel{0^2} - 1$$

$$f'(x) = 3x^2 - 1$$

$$\boxed{f(x) = x^3 - x \quad , \quad f'(x) = 3x^2 - 1}$$





Exercise

① Using $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Show that if $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$ ✓

② Show that if $f(x) = \frac{1-x}{2+x}$, then $f'(x) = \frac{-3}{(2+x)^2}$ ✓

Definition

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

2.7 A function is "differentiable" at 'a' if $f'(a)$ exists.

2.8 It is differentiable on an open interval (a,b) or (a,∞), $(-\infty,a)$, $(-\infty,\infty)$ if it is differentiable at every point in the open interval.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Continuous ✓

differentiable ✓

Conjectures (a statement we do not know to be true)

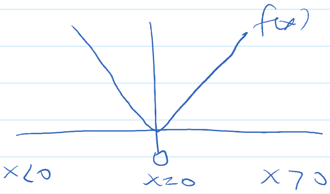
Conjecture 1 — If f is continuous at a point 'a' then f is differentiable at point 'a' ✗

Conjecture 2 — If f is differentiable at a point 'a' then f is continuous at a point 'a' ✓

Exercise

Where is the function $f(x) = |x|$ differentiable?

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



if $x > 0$, $|x| = x$

$x + h > 0$ (for an h small enough)

$f(x+h) = |x+h| = x+h$

$f(x) = |x| = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

if $x < 0$, $|x| = -x$ $\qquad |x+h| = -(x+h)$ for an h small enough

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{x} - h + \cancel{x}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h}$$

$$= \lim_{h \to 0} -1$$

$$= -1$$

for $x = 0$,

$$f(0) = |0| = 0$$

$$f(0+h) = |0+h| = |h|$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|h| - 0}{h}$$

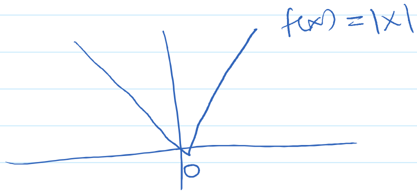
$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

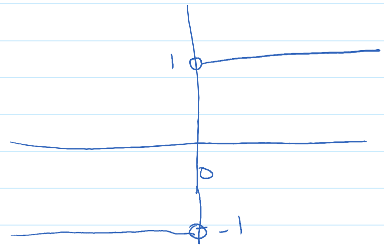
$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1$$

$f'(0)$ does not exist

$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \end{cases}$, $\quad f'(x) = \begin{cases} 1 & \text{if } x > 0 \end{cases}$

$$f(x) = |x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}, \quad f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$





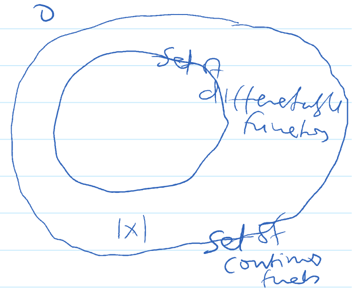
$f(x)=|x|$ is not continuous at '0' because $\lim_{x\to 0} |x| = 1$, $\lim_{x \to 0} |x| = -1$

& not differentiable at 0

Theorem A

If f is differentiable at a point a (p)

then f is continuous at point a (q)



Recall

Logic: If p then q — p, q are statements ($p \Rightarrow q$)

contraposition: If $\sim q$ then $\sim p$ ($\sim q \Rightarrow \sim p$)

Take the contraposition of Theorem A

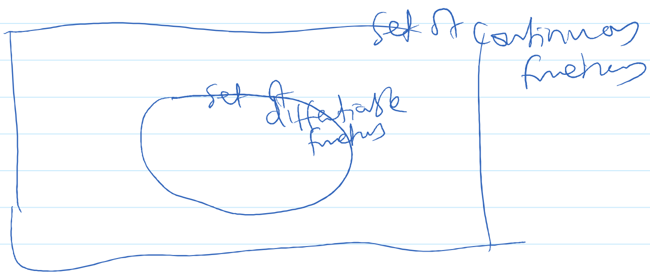
If f is not continuous at a point 'a' ($\sim q$)

then f is not differentiable at a point 'a' ($\sim p$)

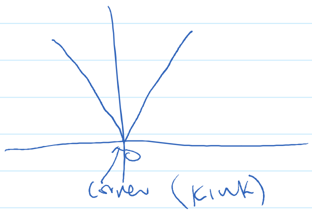
Example

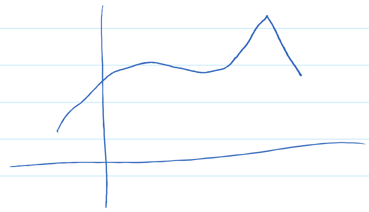
$f(x)=|x|$ is not continuous at a point 'a'

and so $f(x)=|x|$ is not differentiable at 'a'



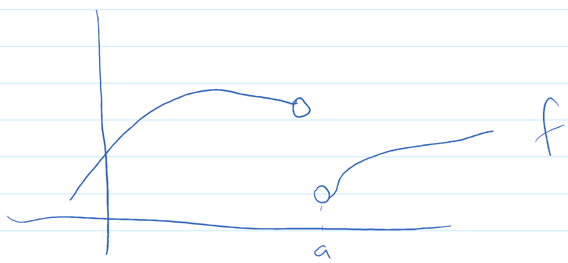
Q How can a function fail to be differentiable ?

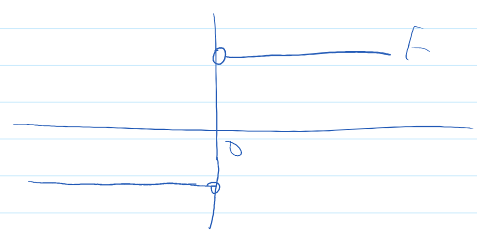




Answer 1 If the function has a kink or corner, then f has no tangent line at the kink or corner so f is not differentiable there.

Answer 2 If there is a point of discontinuity, then the function is not differentiable there





Answer 3 If there is a vertical tangent line, then the function is not differentiable at that point

Higher derivatives

	f'	f'' ([illegible])		$f^{(n)}$ (derivative [illegible])

f	f' (derivative of f) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$	f'' (derivative of f') $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$	$\ldots$	$f^{(n)}$ (derivative of $f^{(n-1)}$) $f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}$

as long as the limits exist

The above notation for the derivative (higher derivatives) are due to Sir Isaac Newton (English mathematician)

Independently both developed calculus
Gottfried Leibniz (German mathematician)

	Newton	Leibniz
$f(x)$ (f a function of x)	$f'(x)$	$\frac{df(x)}{dx}$
	$f''(x)$	$\frac{d}{dx}\left(\frac{df(x)}{dx}\right) = \frac{d^2f(x)}{dx^2}$
	$f'''(x)$	$\frac{d}{dx}\left(\frac{d^2f(x)}{dx^2}\right) = \frac{d^3f(x)}{dx^3}$
	$\vdots$	$\vdots$
	$f^{(n)}(x)$ nth derivative of f	$\frac{d^nf(x)}{dx^n}$

$$f' \Rightarrow \frac{df}{dx} \Rightarrow Df \Rightarrow D_x f$$

Exercise

Recall

Q1 $f(x) = x^3 - x$ $\qquad f'(x) = 3x^2 - 1$ ✓

Exercise

Recall

If $f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$ ✓

$f'(x+h) = 3(x+h)^2 - 1$

find $f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$

$f'(f'(x))$

$$= \lim_{h\to 0} \frac{\left[3(x+h)^2 - 1\right] - (3x^2 - 1)}{h}$$

$$= \lim_{h\to 0} \frac{3(x^2 + 2xh + h^2) - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{3x^2} + 6xh + 3h^2 \cancel{-1} \cancel{-3x^2} \cancel{+1}}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(6x + 3h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 6x + 3h$$

$$f''(x) = 6x$$

$f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

Exercise

$f(x) = x^2$

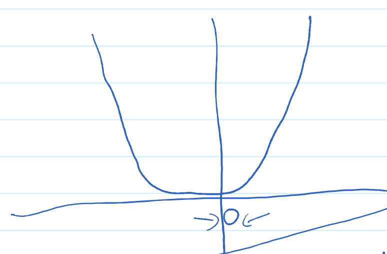
(a) find $f'(0)$ (b) $f'(\frac{1}{2})$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(0+h)^2 - 0^2}{h}$$

$$= \lim_{h\to 0} \frac{h^2 - 0}{h}$$

$$= \lim_{h\to 0} \frac{h^{\cancel{2}}}{\cancel{h}}$$



$\boxed{f \text{ is smooth at } 0}$

$$= \lim_{h\to 0} \frac{\cdot\cdot}{\cancel{h}}$$

$$= \lim_{h\to 0} h$$

$$= 0$$

find (a) $f'(x)$ when $f(x) = x^2$

(b) $f''(x)$

(a)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + h$$

$$= 2x$$

$f(x) = x^2$

$f(x+h) = (x+h)^2$

$f'(x) = 2x$

$f'(x+h) = 2(x+h)$

(b)

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$