2.2 The Limit of a Function

Monday, August 31, 2020   1:13 AM

interval $(0,1)$ is uncountable

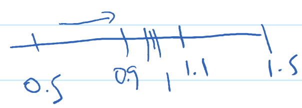
$(0,1) \longmapsto \mathbb{R}$

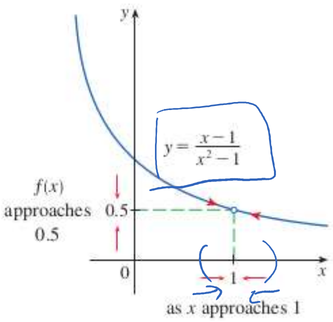
Consider

$$f(x) = \frac{x-1}{x^2-1}$$

Find the limit of $f(x)$ as x get close to 1

Pts getting close to 1 left $(x<1)$	$\frac{x-1}{x^2-1}$	Pts getting Close 1 Right $(x>1)$	$\frac{x-1}{x^2-1}$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
1	0.5	1	0.5





$$\boxed{\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x-1}{x^2-1} = 0.5}$$

$\frac{x-1}{x^2-1}$ approaches 0.5 as x approaches 1

1 Intuitive Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\boxed{\lim_{x\to a} f(x) = L}$$

$f(x) \to L$ as $x \to a$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

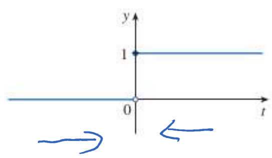
is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

One Sided Limits

The Heaviside function



$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

$$\lim_{t \to 0^-} H(t) = 0$$

limit of $H(t)$ from the left is 0

$$\lim_{t \to 0^+} H(t) = 1$$

lim of $H(t)$ from the right is 1

2 Intuitive Definition of One-Sided Limits

We write

$$\lim_{x \to a^-} f(x) = L$$

$\lim_{x \to a^-} f(x) = L$

and say that the **left-hand limit of $f(x)$** as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

We write

$$\lim_{x \to a^+} f(x) = L$$

$\lim_{x \to a^+} f(x) = L$

and say that the **right-hand limit of $f(x)$** as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.

$\boxed{Q}$ what does it mean to say lim of a function $f(x)$ exist at a point 'a'

$\boxed{A}$ left-hand limit

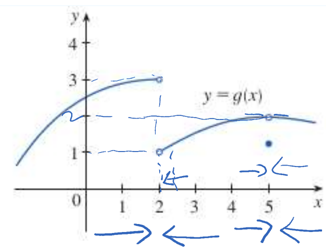
$f(x)$ exist at a point a

Remark

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

$$\boxed{\lim_{x\to a} f(x) = L} \quad \text{limit exists} \quad \overset{\text{Iff}}{\Longleftrightarrow} \quad \boxed{\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L}$$

A) left-hand limit of f at 'a' must agree (equal) with right-hand limit of f at 'a'



Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$

(b) $\lim_{x \to 2^+} g(x)$

(c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$

(e) $\lim_{x \to 5^+} g(x)$

(f) $\lim_{x \to 5} g(x)$

a) $\lim_{x \to 2^-} g(x) = 3$

(limit of x approaching 2 from left)

b) $\lim_{x \to 2^+} g(x) = 1$

(does not exist)

c) $\lim_{x \to 2} g(x) = \text{DNE}$

d) $\lim_{x \to 5^-} g(x) = 2$

e) $\lim_{x \to 5^+} g(x) = 2$

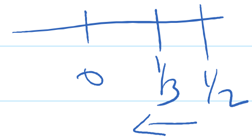
f) $\lim_{x \to 5} g(x) = 2$ (despite $g(5) \neq 2$)

limit of some special function at some point a

1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (despite $\frac{\sin(0)}{0} = \frac{0}{0}$ is indeterminate)

② How can a limit fail to exist

$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$



$f(x) = \sin\frac{\pi}{x}$

$f(0) = \text{DNE}$

$f\left(\frac{1}{2}\right) = \sin\left(\frac{\pi}{1/2}\right) = \sin(2\pi) = 0$ ✓

$f\left(\frac{1}{3}\right) = \sin(3\pi) = 0$

$f\left(\frac{1}{4}\right) = \sin(4\pi) = 0$

$\vdots$

$f\left(\frac{1}{n}\right) = \sin(n\pi) = 0$

$n \geq 1$

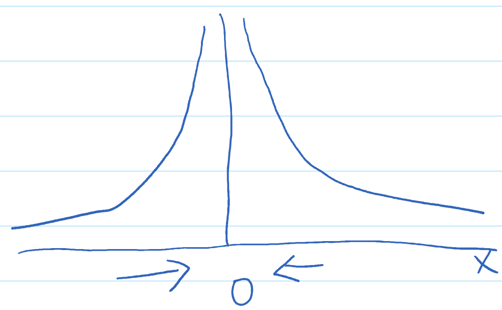
$f\left(\frac{1}{10^6}\right) = 0$

You may guess $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = 0$ ✗

Infinite limit and vertical asymptote

find $\lim_{x \to 0} \frac{1}{x^2}$ if it exist

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
↓	↓
± 0.1	



$$\boxed{\lim_{x \to 0} \frac{1}{x^2} = \infty \quad \text{line } x = 0 \text{ is a vertical asymptote}}$$

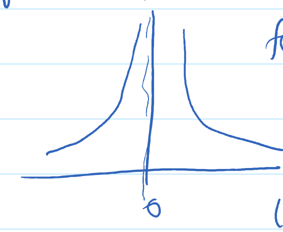
Existence of ... Existence

Existence of infinite limit $\Rightarrow$ Existence of a vertical Asymptotes

Definition

The vertical line $x=a$ is a vertical asymptote of the curve $y=f(x)$ if at least one of ffg is true

Example



$f(x)=\frac{1}{x^2}$

V.A: $x=0$

$\lim_{x\to 0} f(x)=\infty$

(a) $\lim_{x\to a} f(x)=\infty \Rightarrow$ line $x=a$ is a V.A

(b) $\lim_{x\to a^-} f(x)=\infty \Rightarrow$ line $x=a$ is a V.A

(c) $\lim_{x\to a^+} f(x)=\infty \Rightarrow$ line $x=a$ is a V.A

(d) $\lim_{x\to a} f(x)=-\infty \Rightarrow$ line $x=a$ is a V.A

(e) $\lim_{x\to a^-} f(x)=-\infty \Rightarrow$ line $x=a$ is a V.A

(f) $\lim_{x\to a^+} f(x)=-\infty \Rightarrow$ line $x=a$ is a V.A