Review Continuity

Note we defined continuity at a point

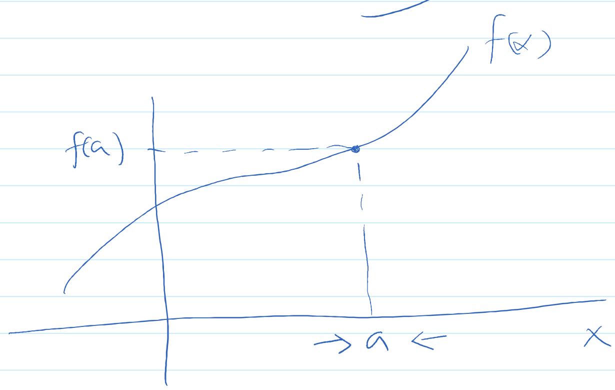
A function f is continuous at a point "a" if

$$\lim_{x \to a} f(x) = f(a)$$

(i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exist

(iii) $\lim_{x \to a} f(x) = f(a)$



Definition 1

A function f is continuous from the right if at a number 'a'

$$\lim_{x \to a^+} f(x) = f(a)$$

Definition 2

A function f is continuous from the left if at a number 'a'

$$\lim_{x \to a^-} f(x) = f(a)$$

Definition 3

A function f is continuous on an interval if it is continuous at every point in the interval.

Polynomial (continuous everywhere $\mathbb{R} = (-\infty, \infty)$) ✓

Rational (continuous on their domain (a subset of $\mathbb{R}$)) ✓

Properties of Continuous functions

1. Let f, g be continuous at a point (a) and let c be a constant

then the following functions are also continuous at 'a'

(a) $f+g$ (d) cf

(b) $f-g$ (e) $\frac{f}{g}$ $(g(a) \neq 0)$

(c) fg

prove that $f+g$ is continuous at 'a', when f, g are both continuous at a

Pf

we know $\lim_{x\to a} f(x) = f(a)$ (since f is continuous at a)

$\lim_{x\to a} g(x) = g(a)$ (since g is continuous at a)

$$\lim_{x\to a} (f+g)(x) = \lim_{x\to a} [f(x) + g(x)]$$ (combination of functions)

$$= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$ (using limit laws)

$$= f(a) + g(a)$$

$$= (f+g)(a)$$

This shows $f+g$ is continuous at a

Polynomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

example

$$f(x) = 5x^2 + 3x + 2$$

They are continuous everywhere $(\mathbb{R} = (-\infty, \infty))$

Rational functions

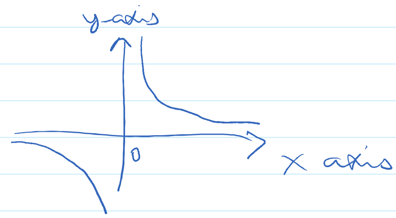
$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P, Q \text{ are polynomial functions}$$

$$(Q(x) \neq 0)$$

Continuous on their domain

Example

$$f(x) = \frac{1}{x}$$



Domain $(-\infty, 0) \cup (0, \infty)$

Exercise

Show that polynomial functions are continuous everywhere

$$f(x) = a_n x^n + a_{n-1} x^n + \cdots + a_1 x + a_0$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$$

$$= \lim_{x \to a} a_n x^n + \lim_{x \to a} a_{n-1} x^{n-1} + \cdots + \lim_{x \to a} a_1 x + \lim_{x \to a} a_0$$

$$= \boxed{a_n} \lim_{x \to a} x^n + \boxed{a_{n-1}} \lim_{x \to a} x^{n-1} + \cdots + \boxed{a_1} \lim_{x \to a} x + \lim_{x \to a} a_0$$

$$= \boxed{a_n} a^n + \boxed{a_{n-1}} a^{n-1} + \cdots + \boxed{a_1} a^1 + a_0$$

in $\mathbb{R}$

we can repeat this steps for any point 'a' in $\mathbb{R}$

we can conclude polynomial functions are continuous on $\mathbb{R}$

Exercise

Do similar for Rational functions

$$f(x) = \frac{P(x)}{Q(x)}, \quad P, Q \text{ are polynomial functions}$$

Conclusion: Rational functions are continuous on their domain.

Theorem

If f is continuous at b $\left(\lim_{x\to b} f(x) = f(b)\right)$

and $\lim_{x\to a} g(x) = b$ (necessary)

then $\lim_{x\to a} f(g(x)) = f(b)$

that is, $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(\cancel{x})\right) = f(b)$

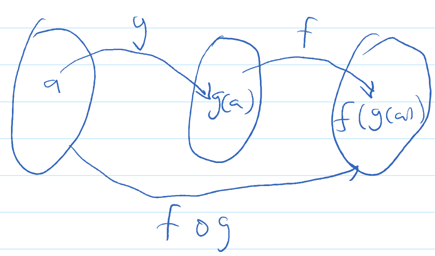
(Passing the limit)

Theorem

If g is continuous at a

and f is continuous at $g(a)$

then $f \circ g$



given by $f\circ g(x) = f(g(x))$ is continuous at 'a'

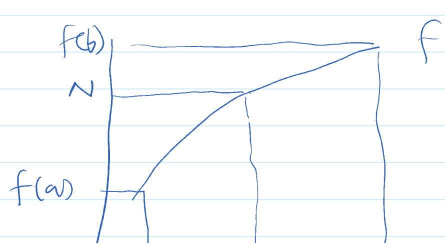
Intermediate Value Theorem

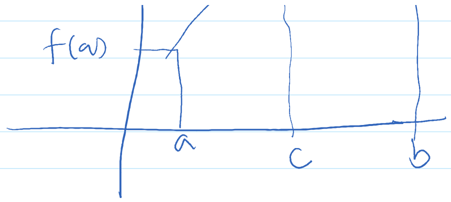
f is continuous on an interval $[a,b]$ (closed interval)

Let N be a number in $[f(a), f(b)]$ $(f(a) \neq f(b))$

There exist a point c in (a,b)

such that $f(c) = N$




f(a)
a
c
b