Exercise on 2.4

use the ε-δ definition to show

$$\lim_{x \to -\frac{3}{2}} \frac{9-4x^2}{3+2x} = 6$$

Solution

given $\varepsilon > 0$, find $\delta > 0$ such that

If $0 < \left|x - \left(\frac{-3}{2}\right)\right| < \delta$ then $\left|\frac{9-4x^2}{3+2x} - 6\right| < \varepsilon$

$0 < \left|x + \frac{3}{2}\right| < \delta$

find δ

use this (known)

$$\left|\frac{9-4x^2}{3+2x} - \frac{6}{1}\right| = \left|\frac{9-4x^2-6(3+2x)}{3+2x}\right| = \left|\frac{9-4x^2-18-12x}{3+2x}\right|$$

$$= \left|\frac{-4x^2-12x-9}{3+2x}\right|$$

$$= \left|\frac{-(3+2x)^{\cancel{2}}}{\cancel{(3+2x)}}\right|$$

$$= \left|-(3+2x)\right|$$

$$= \left|-2\left(\frac{3}{2}+x\right)\right|$$

$$= |-2|\left|\frac{3}{2}+x\right|$$

$$= 2\left|\frac{3}{2}+x\right| < \varepsilon$$

$$\Rightarrow \left|\frac{3}{2}+x\right| < \frac{\varepsilon}{2}$$

$0 < \left|x + \frac{3}{2}\right| < \delta$ ✓

$\boxed{\text{Choose } \delta = \frac{\varepsilon}{2}}$

2.5 Continuity

In Calculus

- Differentiation (tangent problem)
- Integration (Area problem)

limit of a function f as x approach a point 'a'

$$\lim_{x \to a} f(x) = L$$

→ Previously

There are 5 major 'C' words

✓ Continuity
✓ Convergence

Connected
Compactness
complete

↓ beyond this class

$\lim_{x\to a} f(x) = L$

— Connected
Compactness
complete

↓ beyond this class

$\left[\lim_{x\to a} f(x) = f(a) \right]$

Previously $f(x)$ is a polynomial function

Motivate

First, look

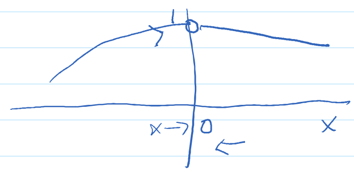
$\lim_{x\to a} f(x) \neq f(a)$

(Limit Exist) but it is not the same as the f at a

Remember $\boxed{f(x) = \frac{\sin x}{x}}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to 0} f(x) \neq f(0)$



$f(x)$ ($f(0)$ is not defined)

In today's class by continuity, we want to discuss when

$\boxed{\lim_{x\to a} f(x) = f(a)}$

Definition

A function f is continuous at a point 'a' if

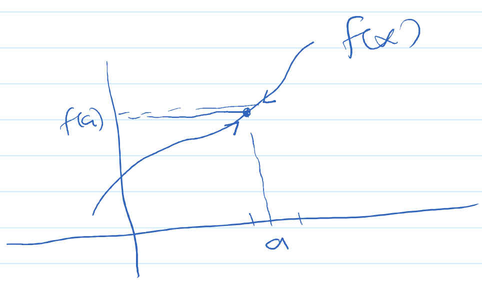
$\lim_{x\to a} f(x) = f(a)$

The ~~defined~~ definition holds if

(i) $f(a)$ is defined

(ii) $\lim_{x\to a} f(x)$ must exist

(iii) $\lim_{x\to a} f(x) = f(a)$



$f(x) \to f(a)$ as $x \to a$

f is continuous at 'a'

Remark

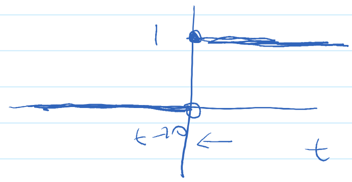
We say f is discontinuous at a
(f has a discontinuity at a)

if f is not continuous at a

Example of a function discontinuous at a point

Heaviside function

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



discontinuous at $t = 0$

$\lim_{t \to 0} H(t)$ DNE

for continuous at a
(i) $f(a)$ defined
(ii) $\lim_{x \to a} f(x)$ exists
(iii) $\lim_{x \to a} f(x) = f(a)$

Some types of discontinuities

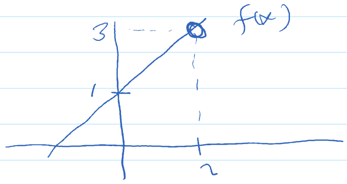
1. Removable discontinuity

consider the ffg examples

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2} = \dfrac{(x-2)(x+1)}{x-2}$

$f(2)$ is not defined

discontinuous at $x = 2$

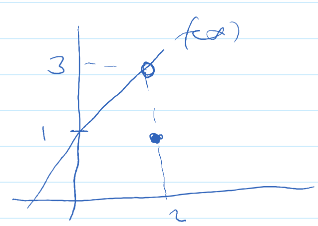


(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

$\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}}$

$= \lim_{x \to 2} (x+1)$

$= 3$



(i) $f(2)$ defined
(ii) $\lim_{x \to 2} f(x)$ exists
(iii) $\lim_{x \to 2} f(x) \neq f(2)$

$3 \neq 1$

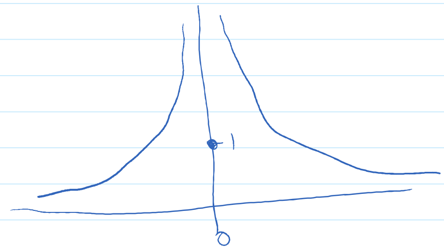
Infinite discontinuity

$\lim x \to 0$

(i) $f(0)$ defined

Infinite discontinuity

$$f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

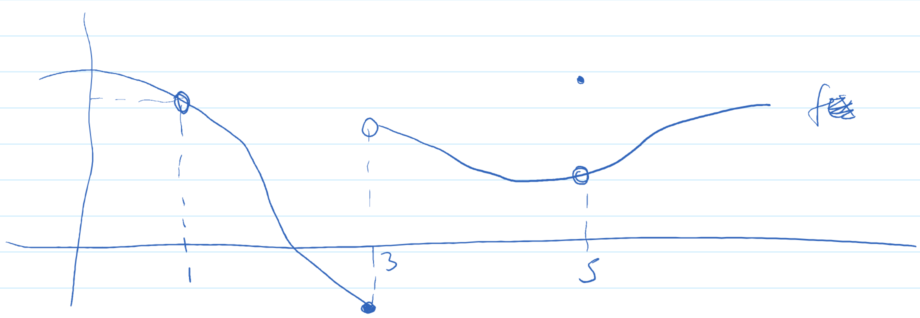


(i) $f(0)$ defined

(ii) $\lim_{x \to 0} f(x) = \infty$ (DNE)

$\boxed{\text{Jump discontinuity}}$ ✓ (see more later).

Example of discontinuity for different reasons



There are 3 discontinuous points on this graph

(i) f is discontinuous at $x=1$ ($f(1)$ is not defined)

(ii) f is discontinuous at $x=3$ (although $f(3)$ is defined $\lim_{x \to 3} f(x)$ does not exist)
DNE

(iii) f is discontinuous at $x=5$ (although $f(5)$ ~~exist~~ is defined
although $\lim_{x \to 5} f(x)$ Exist
But $\lim_{x \to 5} f(x) \neq f(5)$)

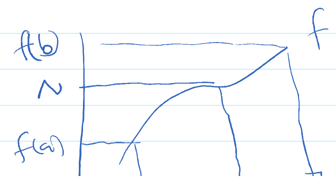
On Monday

Theorems on continuity

"Intermediate value theorem"

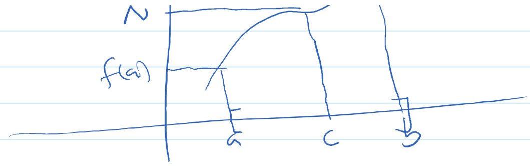
assume f is continuous on $[a,b]$

$f(a) \neq f(b)$



If there exist a point N in $(f(a), f(b))$

$\Rightarrow$ there exist a point c in (a,b)



$\Rightarrow$ there exists a point c in (a,b)