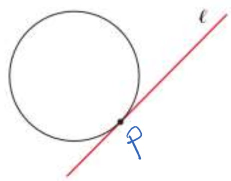
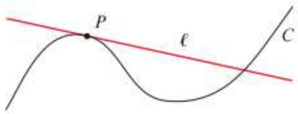
2.1 The Tangent and Velocity Problems

The Tangent Problem



(a)



(b)

Problem Statement

We want to find the equation of the tangent l at the point P

It is a dificult problem to find the equation of a line with one point

Example 1

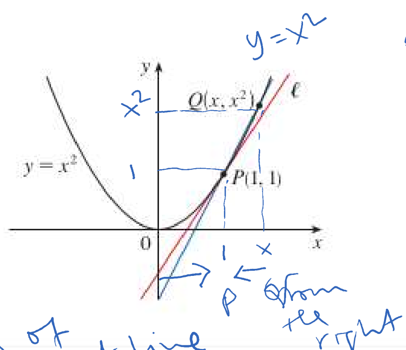
Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Q we want the equation of the tangent ℓ at the point P

Approach

we introduce a new line: line PQ

Q is another point on the curve $y = x^2$



Slope of secant line

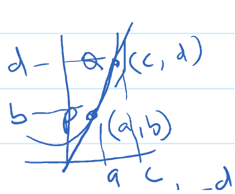
$$m_{PQ} = \frac{x^2 - 1}{x - 1} = \frac{\text{rise}}{\text{run}}$$

What happens to the secant line PQ as Q approaches P

$$\boxed{\text{line } PQ \longrightarrow \text{tangent } \ell \text{ as } Q \longrightarrow P}$$

$$m = \boxed{\lim_{Q \to P} m_{PQ}} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

slope of tangent line



Approaching from right

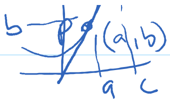
x	m_{PQ} (slope of secant line)
2	3
1.5	2.5
1.1	2.1
1.01	2.01

$$\frac{2^2 - 1}{2 - 1} = \frac{4 - 1}{1} = 3$$

$$\frac{1.5^2 - 1}{1.5 - 1} = \frac{2.25 - 1}{0.5} = 2.5$$

$$1.1^2 - 1 = 1.21 - 1$$

of tangent line



$m_{PQ} = \frac{b-d}{c-a}$

1.1	2.1
1.01	2.01
↓	↓
1	2

$\frac{1.5-1}{} \quad 0.5$

$\frac{1.1^2-1}{1.1-1} = \frac{1.21-1}{0.1} = 2.1$

$\frac{1.01^2-1}{1.01-1} = 2.01$

Approaching from left

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
↓	↓
1	2

$\frac{0^2-1}{0-1} = 1$

$\frac{0.5^2-1}{0.5-1} = 1.5$

$\frac{0.9^2-1}{0.9-1} = 1.9$

$\frac{0.99^2-1}{0.99-1} = 1.99$

$$\boxed{m = \lim_{\substack{x\to 1 \\ Q \to P}} \frac{x^2-1}{x-1} = 2}$$

slope of tangent line

slope of secant line

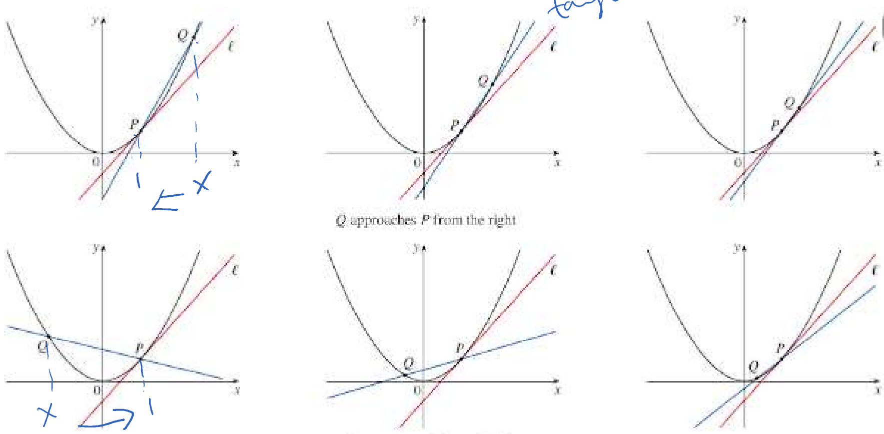
$\boxed{A}$

The equation of the tangent line ℓ to the curve $y=x^2$ at point $P(1,1)$ (x_1, y_1)

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x-1)$

$\boxed{y = 2x - 1}$



Q approaches P from the right

Q approaches P from the left

The Velocity Problem

Find the instantaneous velocity of an object at a specific time (Assuming that you know the position at every other time

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Galileo

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

Galileo.

position of the Ball after 5 sec

$D \propto t^2$

$\boxed{D = 4.9\, t^2}$

$\boxed{S(t) = 4.9t^2}$

Distance

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$ (5+0.001)	49.0049

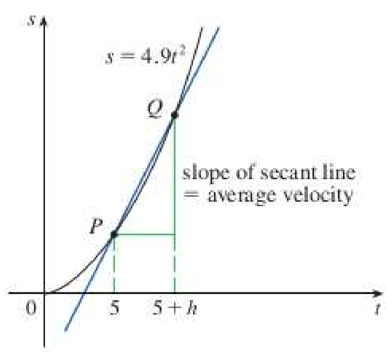
$5 \le t \le (5+h)$ $\quad$ 49

$h \to 0$

$$V = \frac{\text{Change in position}}{\text{Change in time}} = \frac{S(5.1) - S(5)}{5.1 - 5} = \frac{4.9(5.1)^2 - 4.9(5)^2}{5.1-5}$$

$$= 49.49 \text{ m/s}$$

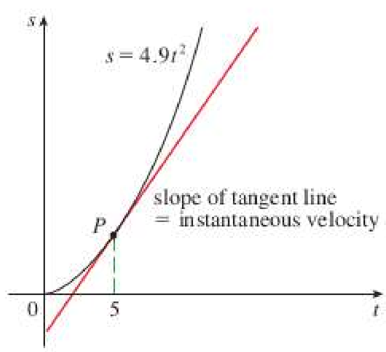
$$\text{Instantaneous velocity} = \lim_{h\to 0} \frac{4.9(5+h)^2 - 4.9(5)^2}{(5+h)-5} = \lim_{h\to 0} \frac{S(5+h) - S(5)}{(5+h)-5}$$



$\boxed{\frac{22}{7}} \to \boxed{\pi}$ — irrational number (non-terminating decimal non repeating patterns)

rational

$\boxed{3.14}$



Class Exercise on 2.1

1. Tangent problem

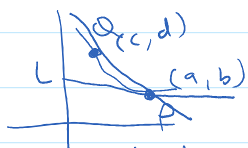
The Point $P(2,-1)$ lies on the curve $y = \frac{1}{1-x}$

(a) If Q is the point $Q\left(x, \frac{1}{1-x}\right)$

Find the slope of the secant line PQ for the following values of x

x	m_{PQ}
1.5	2
1.9	1.11
1.99	1.01
	$\downarrow$ 1

x	m_{PQ}
~~[illegible]~~	0.667
~~[illegible]~~	0.909
	0.99
	$\downarrow$ 1



$m_{PQ} = \frac{d-b}{c-a}$

$Q\left(x, \frac{1}{1-x}\right)$

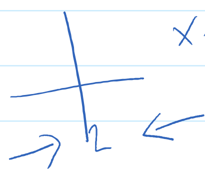
$P(2, -1)$

$$m_{PQ} = \frac{\frac{1}{1-x} - (-1)}{x-2}$$

$x = 1.5$, $= \frac{\frac{1}{1-1.5} + 1}{1.5-2} = \frac{\frac{1}{-0.5} + 1}{-0.5} = 2$

$x = 1.9$, $\frac{\frac{1}{1-1.9} - (-1)}{1.9-2} = \frac{\frac{1}{-0.9} + 1}{-0.1} = 1.11$

$x = 1.99$, $\frac{\frac{1}{1-1.99} - (-1)}{1.99-2} = \frac{\frac{1}{-0.99} + 1}{-0.01} = 1.01$



(b) guess the slope of the tangent line

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to 2} \frac{\frac{1}{1-x} + 1}{x-2} = 1$$

(c) Equation of the tangent line

use point slope formula

$P(2, -1)$, with $x_1 = 2$, $y_1 = -1$

$$y - y_1 = m(x - x_1)$$

$$y - (-1) = 1(x-2)$$

$$y + 1 = x - 2$$

$$y = x - 2 - 1$$

$$\boxed{y = x - 3}$$

equation of the tangent line to the curve at $P(2,-1)$

$y = \frac{1}{1-x}$

Velocity problem

Exercise

If a rock is thrown upward on mars with a velocity of 10 m/s, its height

$\boxed{y = 10t - 1.86t^2}$

(a) find the average velocity over the given time interval

$1 \le t \le 2$

$1 \le t \le 1 + \boxed{1}$

$m_{PQ} = \frac{y(2) - y(1)}{2-1} = \frac{12.56 - 8.14}{1} = 4.42$

$y(2) = 10(2) - 1.86(2)^2 = 12.56$

$y(1) = 10(1) - 1.86(1)^2 = 8.14$

$1 \le t \le 1.5$

$1 \le t \le 1 + \boxed{0.5}$

$y(1.5) = 10(1.5) - 1.86(1.5)^2 = 10.815$

$y(1) = 8.14$

$\frac{y(1.5) - y(1)}{1.5 - 1}$ (1+0.5)

$= \frac{10.815 - 8.14}{1.5 - 1} = \frac{2.675}{0.5} = 5.35$

velocity = $\frac{\text{change Distance}}{\text{time elapsed}}$

$1 \le t \le 1.1$

$1 \le t \le 1 + \boxed{0.1}$

$y(1.1) = 10(1.1) - 1.86(1.1)^2 = 8.7494$

$y(1) = 8.14$

$\frac{y(1.1) - y(1)}{1.1 - 1} = \frac{8.7494 - 8.14}{0.1} = 6.094$ (1+0.1)

$y(1+0.01) =$

$y(1.01) = 10(1.01) - 1.86(1.01)^2 = 8.202614$

$1 \le t \le 1.01$

$1 \le t \le 1 + \boxed{0.01}$

$y(1) = 8.14$

1+0.01

$1 \le t \le 1.01$

$1 \le t \le 1 + \boxed{0.01}$ $\qquad y(1) = 8.14$

$1+0.01$

$$\frac{y(1.01) - y(1)}{1.01 - 1} = \frac{8.202614 - 8.14}{0.01} = \boxed{6.26}$$

$1 + \boxed{0.001}$

$1 \le t \le 1.001$,

b) What is the instantaneous velocity at $t=1$

$$\boxed{m = \lim_{h \to 0} \frac{y(1+h) - y(1)}{(1+h) - 1} =}$$

2.2 Limit of a function

$$f(x) = \frac{x-1}{x^2-1}$$

$$\boxed{\lim_{x \to 1} \frac{x-1}{x^2-1}}$$

Numerical

$(x<1)$ x	$f(x)$	$(x>1)$ x	$f(x)$
0.5		1.5	
0.9		1.1	
0.99		1.01	
0.999		1.001	

graph

$$f(x) = \frac{x-1}{x^2-1}$$

(what is $f(x)$ for x values near 1)

Show $\displaystyle\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

Confirm (using similar approach described above) numerical & graphical

Numerical & graphical

$$\lim_{x \to 0} \frac{\sqrt{x^2+9} - 3}{x^2} = \frac{1}{6}$$