2.4 Precise definition of a limit

[There is a $\boxed{\text{delta}}$ for every $\boxed{\text{epsilon}}$]

(given an $\varepsilon > 0$
your task is to find a $\underline{\delta > 0}$)

$\varepsilon - \delta$

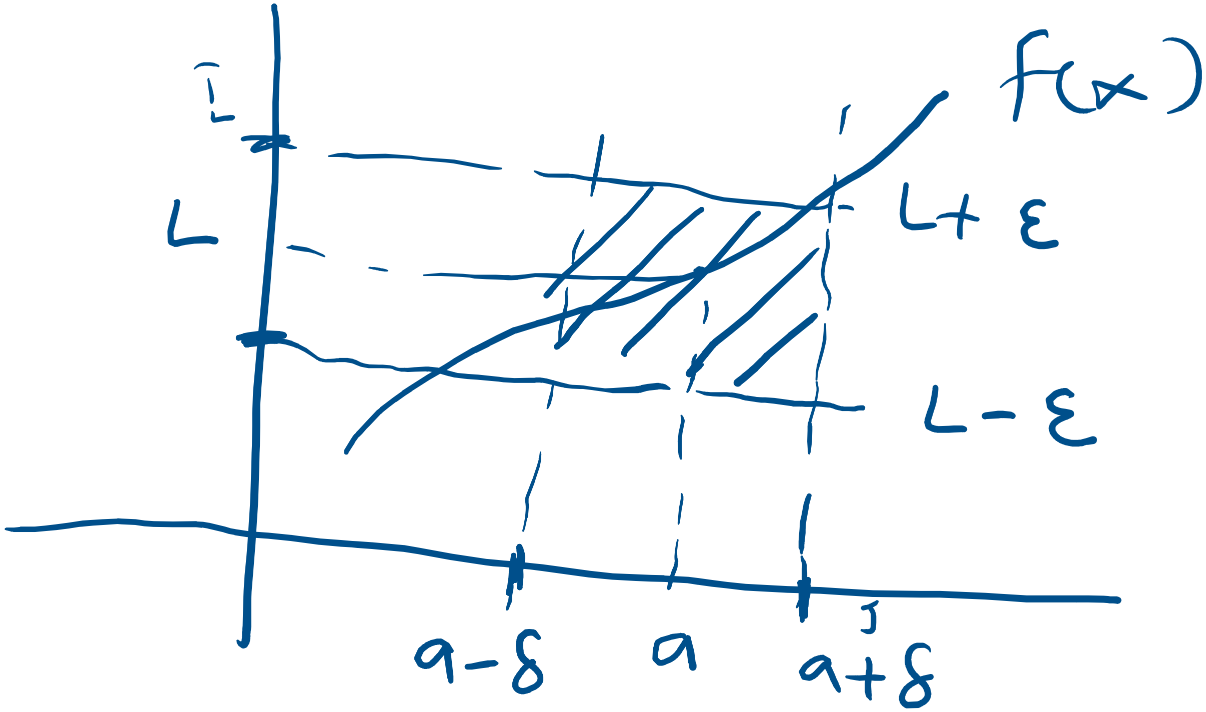
Epsilon-delta definition of the limit

given an epsilon ($\varepsilon > 0$)

there exist a delta ($\delta > 0$)

such that

if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$



f(x)
L+ε
L
L−ε
a−δ
a
a+δ

Interval definition of a limit

for every open interval I around L

there exist an open interval J around a

such that if x is in J

then f(x) is in I

Exercise

use the $\varepsilon-\delta$ definition

$$\lim_{x \to 3} (4x-5) = 7$$

Solution

given $\varepsilon > 0$, find $\delta > 0$

if $0 < |x-3| < \delta$ then $\boxed{|(4x-5)-7| < \varepsilon}$

if $0 < |x-3| < \delta$ then $4|x-3| < \varepsilon$

If $0 < |x-3| < \delta$ then $4|x-3| < \varepsilon$ ✓

$|x-3| < \boxed{\frac{\varepsilon}{4}}$

take $\delta = \frac{\varepsilon}{4}$

(There is a δ for every ε)

$$
\begin{aligned}
|(4x-5)-7| &= |4x-12| \\
&= 4|x-3| \\
&< 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon
\end{aligned}
$$

$$
\begin{aligned}
|(4x-5)-7| &= |4x-5-7| \\
&= |4x-12| \\
&= |4(x-3)| \\
&= |4|\,|x-3| \\
&= 4\,|x-3| < \varepsilon
\end{aligned}
$$

Recall

$$
\begin{aligned}
|c\,f(x)| &= |c|\,|f(x)| \quad , c > 0 \\
&= c\,|f(x)|
\end{aligned}
$$

Exercise 2

prove that $\lim_{x \to 3} x^2 = 9$

use $\varepsilon - \delta$ definition

Solution

given $\varepsilon > 0$, find $\delta > 0$

if $0 < |x-3| < \delta$ then $|x^2 - 9| < \varepsilon$

$$|x^2-9| = |x^2-3^2|$$
$$= |(x-3)(x+3)|$$
$$= |x-3||x+3| < \varepsilon$$

If $0<|x-3|<\delta$ then $|x-3||x+3| < \varepsilon$

we can choose a positive constant C such that $|x+3| < C$

$$|x-3||x+3| < c|x-3|$$

$$< \varepsilon$$

(Choose a c such that $|x+3| < c$

$$c|x-3| < \varepsilon$$

$$|x-3| < \frac{\varepsilon}{c}$$

Choose $\delta = \dfrac{\varepsilon}{c}$

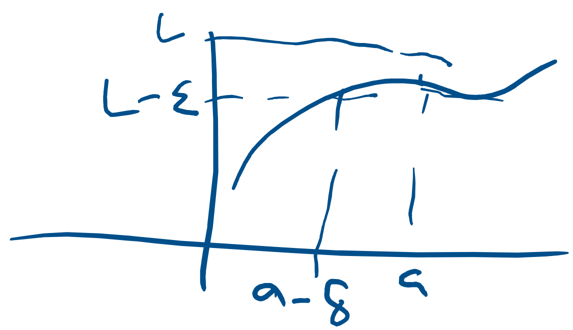
One-sided limit

$$\lim_{x \to a^-} f(x) = L$$

given $\varepsilon > 0$ there is a $\delta > 0$

such that

if $a - \delta < x < a$ then $|f(x) - L| < \varepsilon$



$$\lim_{x \to a^+} f(x) = L$$

given $\varepsilon > 0$ there is a $\delta > 0$

such that

If $a < x < a + \delta$ then $|f(x) - L| < \varepsilon$

Prove $\lim_{x \to 0^+} \sqrt{x} = 0$

given $\varepsilon > 0$, find $\delta > 0$

if $0 < x < \delta$ then $|\sqrt{x} - 0| < \varepsilon$

if $0 < x < \delta$ then $|\sqrt{x}| < \varepsilon$

if $0 < x < \delta$ then $\sqrt{x} < \varepsilon$

$$\sqrt{x} \le \varepsilon$$

$$(\sqrt{x})^2 < \varepsilon^2$$

$$x < \varepsilon^2$$

choose $\delta = \varepsilon^2$ ✓

Prove that

if $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$

$$\lim_{x \to a} [f(x) + g(x)] = L + M$$

Pf

given $\varepsilon > 0$ find a $\delta > 0$

if $0 < |x - a| < \delta$ then $|(f(x) + g(x)) - (L+M)| < \varepsilon$

$$|(f(x)+g(x)) - (L+m)| = |(f(x)-L) + (g(x)-m)|$$

$$\le |f(x)-L| + |g(x)-m|$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

Recall

$$|x+y| \le \overset{\substack{\text{House}\\ \downarrow}}{|x|} + \overset{\substack{\text{Car}\\ \downarrow}}{|y|} \quad (\text{triangle inequality})$$

we have a $\delta_1 > 0$ such that

If $0 < |x-a| < \delta_1$ then $|f(x) - L| < \frac{\varepsilon}{2}$

we have a $\delta_2 > 0$ such that

If $0 < |x-a| < \delta_2$ then $|g(x) - L| < \frac{\varepsilon}{2}$

choose $\delta = \min\{\delta_1, \delta_2\}$

If $0 < |x-a| < \delta$ then $|(f(x)+g(x)) - (L+m)| < \varepsilon$