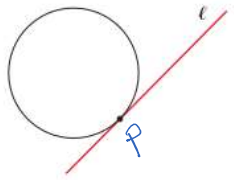
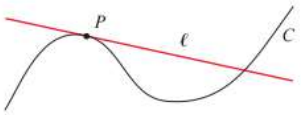
2.1 The Tangent and Velocity Problems

The Tangent Problem



(a)



(b)

Problem Statement

We want to find the equation of the tangent l at the point P

It is a difficult problem to find the equation of a line with one point

Example 1

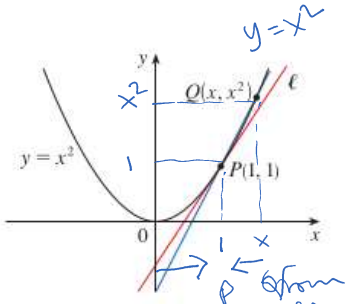
Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Q we want the equation of the tangent ℓ at the point P

Approach

we introduce a new line: line PQ

Q is another point on the curve $y = x^2$



What happens to the secant line PQ as Q approaches P

slope of secant line

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

= rise / run

line PQ → tangent ℓ as Q → P

$$m = \lim_{Q \to P} m_{PQ} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

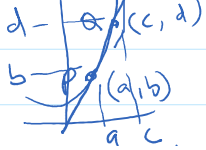
slope of tangent line

Approaching from right

x	m_{PQ} (slope of secant line)
2	3
1.5	2.5
1.1	2.1

$$\frac{2^2 - 1}{2 - 1} = \frac{4 - 1}{1} = 3$$

$$\frac{1.5^2 - 1}{1.5 - 1} = \frac{2.25 - 1}{0.5} = 2.5$$



of tangent line

Approaching from left

x	m_{PQ}
0	1
0.5	1.5
0.9	1.5
0.99	1.99
↓	↓
1	2

$m_{PQ} = \frac{b-d}{c-a}$

1.1	2.1
1.01	2.01
↓	↓
1	2

$\frac{0^2-1}{0-1} = 1$

$\frac{0.5^2-1}{0.5-1} = 1.5$

$\frac{0.9^2-1}{0.9-1} = 1.9$

$\frac{0.99^2-1}{0.99-1} = 1.99$

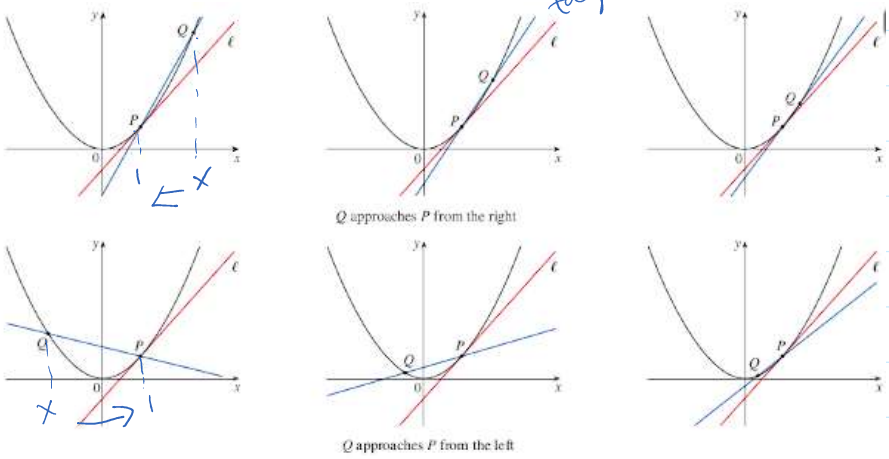
$\frac{1.5^2-1}{1.5-1} = \frac{1.25}{0.5}$

$\frac{1.1^2-1}{1.1-1} = \frac{1.21-1}{0.1} = 2.1$

$\frac{1.01^2-1}{1.01-1} = 2.01$

$m = \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$ (A)

slope of tangent line — slope of secant line



The equation of the tangent line ℓ to the curve $y=x^2$ at point $P(1,1)$ (x_1, y_1)

$y - y_1 = m(x - x_1)$

$y - 1 = 2(x-1)$

$y = 2x - 1$

The Velocity Problem

Find the instantaneous velocity of an object at a specific time

(Assuming that you know the position at every other time

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

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Galileo.

$D \propto t^2$

$D = 4.9t^2$

$S(t) = 4.9t^2$

Distance

Time interval	Average velocity (m/s)
$5 \leq t \leq 5.1$	49.49
$5 \leq t \leq 5.05$	49.245
$5 \leq t \leq 5.01$	49.049
$5 \leq t \leq 5.001$ (5 to 5.001)	49.0049

$5 \leq t \leq (5+h)$, $h \to 0$ ↓ 49

$v = \frac{\text{change in position}}{\text{change in time}} = \frac{S(5.1) - S(5)}{5.1 - 5} = \frac{4.9(5.1)^2 - 4.9(5)^2}{5.1 - 5}$

$= 49.49$ m/s

Instantaneous velocity $= \lim_{h \to 0} \frac{4.9(5+h)^2 - 4.9(5)^2}{(5+h) - 5}$

$\frac{22}{7}$ → π — irrational number

(non-terminating decimal, non repeating patterns)

↑ rational

3.14