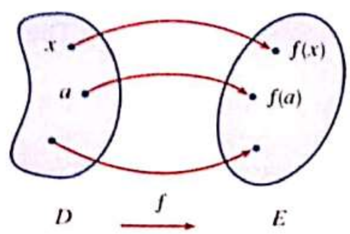
Function Review

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.



Examples

$\boxed{f(x) = x}$

$f(1) = 1$

$f(2) = 2$

⋮

Domain: all real numbers

$(-\infty, \infty)$

$(-\infty, 0)\ (0, \infty)$

$f(x) = x^2$

$f(1) = 1$

$f(2) = 4$

$f(3) = 9$

Domain: all real

$f(x) = \frac{1}{x}$

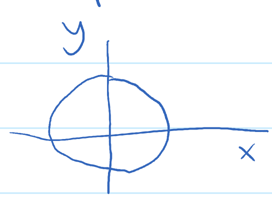
Domain: all real except 0

$(-\infty, 0) \cup (0, \infty)$

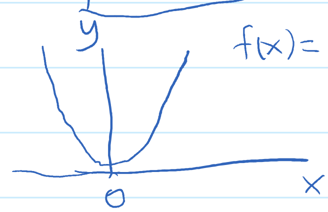
~~The Vertical Line Test~~ A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

not a function

Equation vs function



$x^2 + y^2 = 1$



$f(x) = x^2$

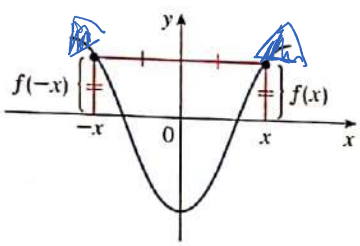


FIGURE 19
An even function

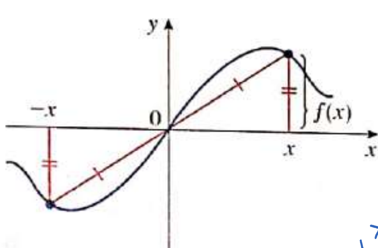


FIGURE 20
An odd function

Even functions

$f(-x) = f(x)$

Example

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

$f(x) = x^4$

$f(x) = x^6$

odd functions

$f(-x) = -f(x)$

Example

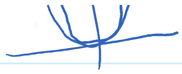
$f(x) = x^3$

$f(-x) = (-x)^3$

$= (-x)\cdot(-x)\cdot(-x)$

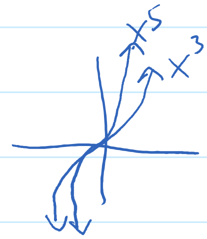
$= -x^3$

An odd function



$f(x) = x^6$

$= (-x)\cdot(-x)\cdot(-x)$
$= -x^3$
$= -f(x)$

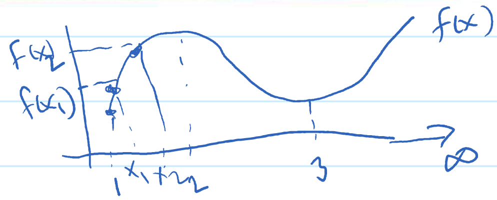


A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$

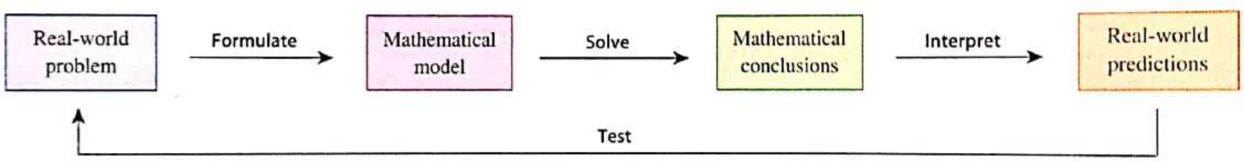
It is called **decreasing** on I if

$$f(x_1) > f(x_2) \quad \text{whenever } x_1 < x_2 \text{ in } I$$



Increasing on $[1,2] \cup [3,\infty)$

Decreasing on $[2,3]$



when we use functions to study real life problem, we call them mathematical models

Linear Models

When we say that y is a **linear function** of x, we mean that the graph of the function is a line, so we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = \boxed{mx^1} + b$$

leading term

← linear, leading term has exponent = 1

quadratic
$y = f(x) = ax^2 + bx + c$
$(a \neq 0)$

Polynomials

A function P is called a **polynomial** if

$$P(x) = \boxed{a_n x^n} + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are constants called the **coefficients** of the polynomial. The domain of any polynomial is $\mathbb{R} = (-\infty, \infty)$. If the **leading coefficient** $a_n \neq 0$, then the **degree** of the polynomial is n. For example, the function

$$P(x) = 2x^6 - x^4 + \tfrac{2}{5}x^3 + \sqrt{2}$$

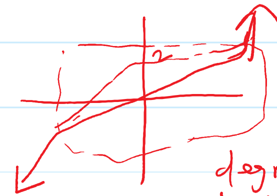
is a polynomial of degree 6.

Graph
$f(x) = \boxed{x^4} + 2$
even

is a polynomial of degree 6.

$f(x) = \boxed{x^5} + 2$ ~~$+ \cancel{3x^2 + 4}$~~

odd



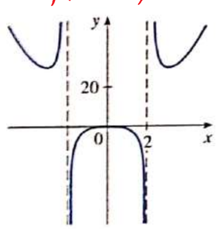


FIGURE 18
$f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$

$x=-2$ $x=2$

Dom
$(-\infty,-2) \cup (-2,2) \cup (2,\infty)$

V.A: $x=-2$
$x=2$

H.A: None

Rational Functions

A **rational function** f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain consists of all values of x such that $Q(x) \neq 0$. A simple example of a rational function is the function $f(x) = 1/x$, whose domain is $\{x \mid x \neq 0\}$; this is the reciprocal function graphed in Figure 14. The function

$$f(x) = \frac{2x^4 - x^2 + 1}{x^2 - 4}$$

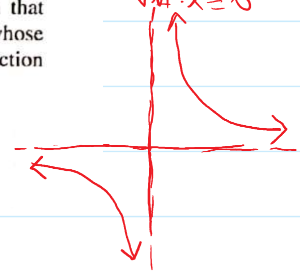
is a rational function with domain $\{x \mid x \neq \pm 2\}$. Its graph is shown in Figure 18.

$1 = x^0$

degree num=0
~~degree~~ denom=1

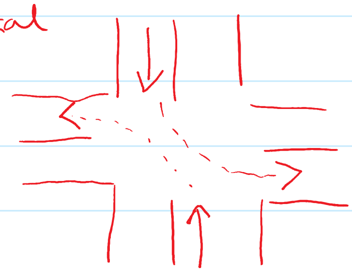
$\boxed{f(x) = \dfrac{1}{x^1}}$

V.A: $x=0$



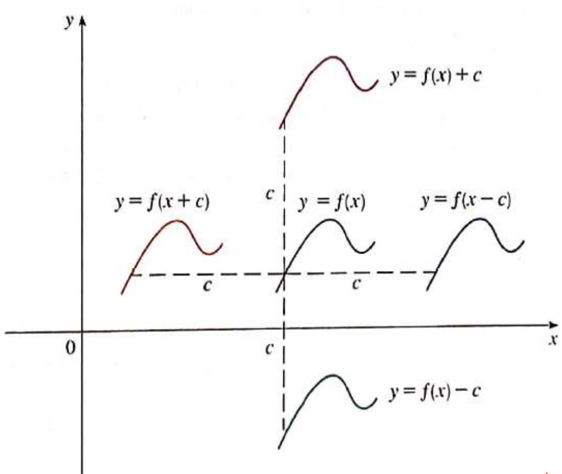
Domain
$(-\infty,0) \cup (0,\infty)$
Vertical asymptote line $x=0$

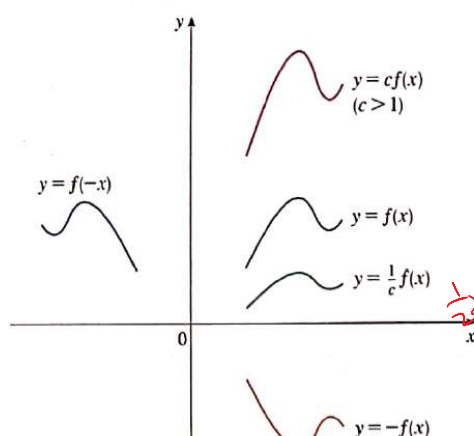
degree of a polynomial
the highest exponent
Horizontal asymptote
H.A: $y=0$ $(0<1)$



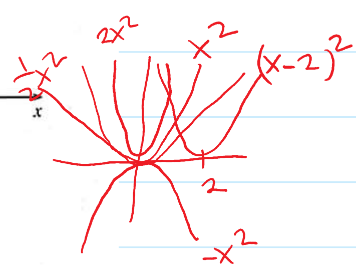
Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left





$r(x) = 2x^2$
$f(x) = x^2$
$h(x) = -x^2$
$g(x) = (x-2)^2$

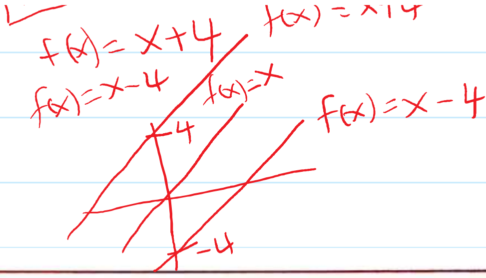


$f(x) = x$ linear function

$f(x) = x+4$ $f(x) = x+4$
$f(x) = x-4$ $f(x) = x$

$\boxed{f(x) = x^2}$

$g(x) = 2(x-3)^2 + 4$



$g(x) = 2(x-3)^2 + 4$

Vertical stretch — Horizontal shift right — vertical shift up

Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c

$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c

$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c

$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c

$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis

$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis

Combinations of Functions

Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg, and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

Definition Given two functions f and g, the **sum**, **difference**, **product**, and **quotient** functions are defined by

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)\,g(x) \qquad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

f (You know the domain of f

g (Dom g is known

$f+g$ (on intersection of Dom$_f$ and Dom$_g$

$(f+g)(x) = f(x) + g(x)$ on Dom$_f \cap$ Dom$_g$

$(f-g)(x) = f(x) - g(x)$ on Dom$_f \cap$ Dom$_g$

$(fg)(x) = f(x) \cdot g(x)$ on Dom$_f \cap$ Dom$_g$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ on Dom$_f \cap$ Dom$_g$, $g(x) \neq 0$

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$

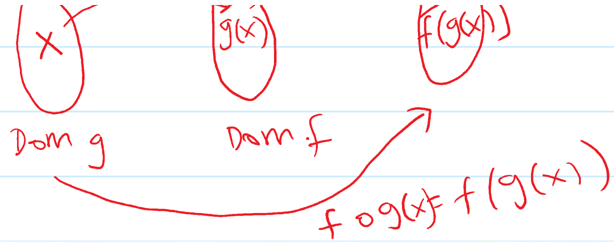
$\boxed{\text{Dom}_f \subseteq \text{Range}_g}$

$f(x) = \sqrt{x}$

Dom f: $[0, \infty)$

$f(x) = \frac{1}{\sqrt{x}}$

Dom f ...



Dom$_f$ = Range$_g$

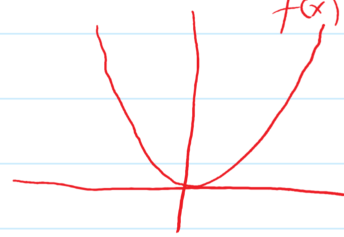
$\sqrt{x}$

Dom f : $(0, \infty)$

> **1 Definition** A function f is called a **one-to-one function** if it never takes on the same value twice; that is,
>
> $$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

Horizontal line intersect a 1-1 function at exactly one point

$f(x) = x^2$



> **Horizontal Line Test** A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Dom$_f$: $(-\infty, \infty)$

This function is not 1-1

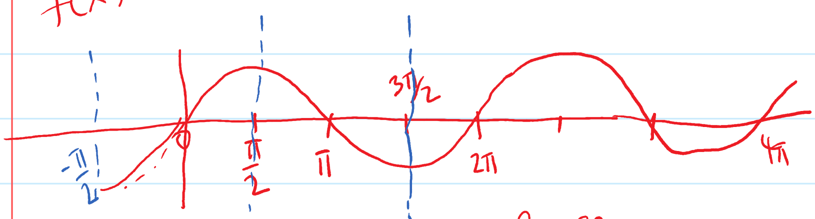
Q Can I make it 1-1

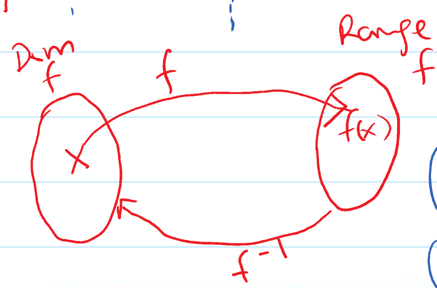
A restrict the domain

$f(x) = x^2$ on $[0, \infty)$

Consider Trigonometric functions

$f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$





$(f \circ f^{-1})(x) = x$

$(f^{-1} \circ f)(x) = x$

Inverse reciprocal

$$\boxed{f^{-1}(x) = \frac{1}{f(x)}} \quad \times$$

$$\boxed{3^{-1} = \frac{1}{3}} \quad \checkmark$$

> **2 Definition** Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by
>
> $$f^{-1}(y) = x \iff f(x) = y$$
>
> for any y in B.

domain of f^{-1} = range of f
range of f^{-1} = domain of f

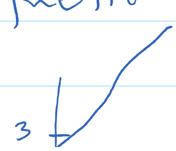
$f^{-1}(x)$ does *not* mean $\dfrac{1}{f(x)}$

$(f \circ g)(x) = (g \circ f)(x)$ only if $g = f^{-1}$

find Inverse of the following functions

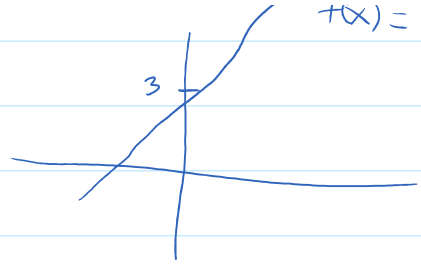
$$\boxed{f(x) = 2x + 3}$$

$f(x) = 2x + 3$



$$\boxed{f(x) = 2x + 3}$$

1. pick a number x
2. multiply by 2
3. add 3



To find Inverse

1. pick a number x
2. subtract 3
3. divide by 2

$$\boxed{f^{-1}(x) = \frac{x-3}{2}}$$

Exercise

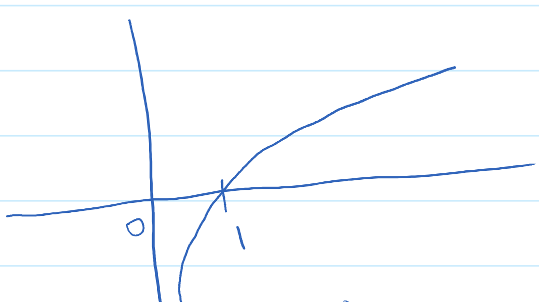
$f(x) = 2x+3$

$f^{-1}(x) = \frac{x-3}{2}$

$$\begin{aligned}(f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right) + 3 \\ &= x - 3 + 3 = x\end{aligned}$$

Logarithmic functions

$f(x) = \log(x)$



Dom $(0, \infty)$

Exponential functions

$f(x) = a^x$

growth

$a > 1$

Dom $(-\infty, \infty)$

Range $(0, \infty)$

decay

$0 < a < 1$

Dom " $(0, \infty)$

Range $(-\infty, \infty)$

$\log(\frac{1}{2}) = -ve$

$0<a<1$

$\log_{10} 1 = 0$

$\log_{10} 10 = 1$

$\log_{10} 100 = 2$

$\log_{10} 1000 = 3$

$\log_{10} 1000 = 4$

laws of logarithms

$$\log(AB) = \log A + \log B$$

$$\log\left(\frac{A}{B}\right) = \log A - \log B$$

$$\log A^n = n \log A$$

$$\log_A A = 1$$

$$\log_B A = \frac{\log_C A}{\log_C B}$$

$$\log_2 5 = \frac{\log 5}{\log 2}$$

expand

$$\log_2 \sqrt{\frac{x^2+1}{x-1}} = \log_2 \left(\frac{x^2+1}{x-1}\right)^{1/2}$$

$$= \frac{1}{2} \log_2 \frac{x^2+1}{x-1}$$

$$= \frac{1}{2}\left(\log_2 (x^2+1) - \log_2 (x-1)\right)$$

$$\sqrt{x} = x^{1/2}$$

$$\sqrt[3]{x} = x^{1/3}$$

$$\sqrt[n]{x} = x^{1/n}$$

$$\sqrt{4} = 4^{1/2} = 2$$

Express as a single logarithm

$$\frac{1}{2} \ln x - 2 \ln(x^2+1)$$

$\ln x = \log_e x$

e = Euler

$\frac{1}{2}$

$$\frac{1}{2}\log_e x - 2\log_e(x^2+1)$$

$$\ln x^{1/2} - \ln(x^2+1)^2$$

$$\ln \frac{x^{1/2}}{(x^2+1)^2}$$

$$\ln \frac{x^{1/2}}{(x^2+1)^2}$$

$i = \sqrt{-1}$

e = Euler number

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \quad 2.7 \qquad \left(1+\frac{1}{2}\right)^2}$$

The most beautiful equation in math

$$\boxed{e^{i\pi} = -1}$$

$$f(x) = e^x$$

$$\frac{d f(x)}{dx} = e^x$$